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平板內賓漢流體之暫態流動型態受已知入口流量條件之影響

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行政院國家科學委員會補助專題研究計畫 ■ 成 果 報 告
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中　華　民　國　94　年　10　月　30　日

平板內賓漢流體之暫態流動型態受已知入口流量條件之影響

Unsteady unidirectional flow of Bingham fluid between parallel plates with different given volume flow rate conditions

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中文摘要

本研究為探討平板間，賓漢流體流動受已知入口流量條件之影響時,非穩態速度與壓力梯度分佈之情形。於平板入口施加已知且隨時間改變的體積流率，造成平板間流動情形之改變，根據此已知之入口條件，首先解出當條件分別為突然起動與等加速度之基本流動解，再將其結果應用於模擬實際情形之活塞運動中，首先此活塞先等加速至某一定速後，保持定速度一段時間，再等減速至停止，此活塞運動本文稱為梯形運動。

關鍵字：賓漢流體，速度分佈，拉式轉換，梯形運動

Abstract

In this paper, the velocity profile and pressure gradient of the unsteady state unidirectional flow of Bingham fluid between parallel plates are solved by the Laplace transform method. The flow motion between the plates is induced by a prescribed arbitrary inlet volume flow rate which varies with time. Based on the flow conditions described, two basic flow situations are solved; these are a suddenly started, and a constant acceleration, flow respectively. These two results are then applied to a practical case that is a trapezoidal piston motion which contains three phases of piston motion, the constant acceleration from the rest to a fixed velocity, then keeping at this velocity, following with the constant deceleration to a stop. In addition, oscillatory flow is also considered. The result indicates when the yield stress τ_0 is equal to zero, the solution to the problem reduces to that of a Newtonian fluid.

Keywords: Bingham fluid, velocity profile, pressure gradient, Laplace Transform, Trapezoidal piston motion

1. Introduction

A great deal of literature deals with the solution of the various types of fluids flowing internally. Among the fluids, the Newtonian fluid is the simplest to solve, not only numerically but also analytically. However, its application is limited, as very few liquids obey the Newtonian law. In practice, such as chemical, petroleum, mechanical, material processing and nuclear industries, the behavior of several fluids greatly deviates from Newtonian fluid. Based on the knowledge of solutions to a Newtonian fluid, the different fluids can be examined, such as Maxwell fluid, Voigt fluid, Oldroyd-B fluid, Bingham fluid, Rivlin-Ericksen fluid or power-law fluid. In this study, we investigate the flow characteristics of Bingham fluid. Due to many industrial important materials, including concentrated suspensions of all types, do not flow until a certain critical (or yield) shear stress τ_0 is exceeded. The region where the shear stress is less than this prescribed yield stress, a velocity gradient will not exist, that means the velocity within this area will be all the same. This is so-called “Plug flow” region.

The exact solutions for laminar flow in a duct with a given pressure gradient varying with time are well known. For example, Szymanski [1] solved the impulsively imposed pressure gradient and Rahaman et al. [2] treated Maxwell fluid for the pressure gradient varying exponentially with time, sinusoidal pulsating pressure gradient and constant pressure gradient. Some researchers studied the flow motion caused by a moving boundary. Bandelli et al. [3] solved various start-up flows of Maxwell fluids in domains with one finite dimension by integral transform method. Huang et al. [4] solved the analytic solution and investigated the character of viscoelastic fluids in a double-gap concentric cylinder rheometer.

Generally, in realistic applications, the inlet volume flow rate is the given condition instead of the pressure gradient. Pascal et al. [5] solved the power-law fluid with a given volume flow rate which varies with time by similarity transformation method. Das et al. [6] solved the problem using the Newtonian fluid with various types of given inlet piston motion in the channel and duct analytically. Also Das et al. [7] verified their results experimentally. However, how non-Newtonian fluid affect the flow pattern under this given volume flow rate condition is left to be found out; therefore, this is the reason intrigues us conducting a series of studies solving the same problem with different fluids.

2. Mathematical Formulation

The rheological equation of state for a Bingham fluid is given by Lee et.al.[8] as

$$\vec{\vec{T}} = -p\,\vec{\vec{I}} + \vec{\vec{\tau}}\,, \tag{1}$$

and $$\vec{\vec{\tau}} = \vec{\vec{\tau}}_0 + 2\mu\,\vec{\vec{D}} \tag{2}$$

where $\vec{\vec{T}}$ is the stress tensor, p is the static fluid pressure (p=p(x,y,z)), $\vec{\vec{I}}$ is the identity, $\vec{\vec{\tau}}$ is the extra stress tensor, $\vec{\vec{\tau}}_0$ is the yield stress tensor, μ is the viscosity coefficient and assumed to be constant, $\vec{\vec{D}}$ is the deformation tensor.

The problem of the transient flow of incompressible Bingham fluid between parallel plates is considered. The dynamic equation is

$$\nabla\cdot\vec{\vec{T}} + \rho\,\vec{b} = \rho\frac{d\vec{V}}{dt}\,. \tag{3}$$

In the above equation, $\vec{V}$ denotes the velocity vector, $\vec{b}$ the body force field and $\nabla\cdot$ the divergence operator.

The continuity equation is

$$\nabla\cdot\vec{V} = 0\,. \tag{4}$$

Using the Cartesian coordinate system (x, y, z), the x-axis is taken as the centerline direction between these two parallel plates, y is the coordinate normal to the plate, z is the coordinate normal to x and y, respectively. The velocity field and stress are assumed in the form

$$\vec{V} = u(y,t)\,\vec{i}\,, \tag{5}$$

$$\vec{\vec{\tau}} = \tau_{yx}\,\vec{i}\,, \tag{6}$$

where u is the velocity in the x-coordinate direction, τ_{yx} is the stress which acts on the y-plane toward x direction, $\vec{i}$ is the unit vector in the x-coordinate direction. This effectively assumes that the flow is fully developed at all points in times.

Substitution of Eq. (5) into Eq.(4) shows that the continuity equation is automatically satisfied while substitution into Eq. (2) and (3) yields the following scalar equations:

$$\tau_{yx} = \tau_0 + \mu\frac{\partial u}{\partial y}\,, \tag{7}$$

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_{yx}}{\partial y} - \rho\frac{\partial u}{\partial t}\,, \tag{8}$$

where subscript y denotes the plane normal to y direction, x the direction along the shear stress, and

$$\frac{\partial p}{\partial y}=\frac{\partial p}{\partial z}=0, \quad (9)$$

where the body force is incorporated into the term of pressure gradient.

Eq. (8) and (9) implies that the pressure gradient is a function of time only.

Substituting Eq. (7) into Eq. (8) yields the equation of motion in the *x*-direction as follows:

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \quad (10)$$

3. Methodology of Solution

Since the governing equation, boundary conditions and initial condition are known, the problem is well posed. In general, it is not convenient to solve this equation by the method of separation of variables or eigenfunction expansion. In this study, the Laplace transform method is used to reduce the two variables into single variable. This procedure greatly reduces the difficulties of treating the original partial differential equation.

As these two plates are $2h$ apart; therefore, the boundary conditions are

$$u(h,t)=0, \quad (11)$$

and $$\frac{\partial u(h_0,t)}{\partial y}=0. \quad (12)$$

h_0 is the distance from the centerline where the fluid is subject to yielding stress τ_0 and begins to deform.

The initial state is assumed to be quiescent; therefore, the initial condition is $u(h,0)=0$. However the problem can be solved if the pressure gradient function is known. In this study, the pressure gradient is determined indirectly by the inlet volume flow rate, which is prescribed. The velocity is related to the volume flow rate by

$$\int_0^{h_0}u_{\max}(h_0,t)dy+\int_{h_0}^{h}u(y,t)dy=u_p(t)h=Q(t)/2 \quad (13)$$

where $u_{\max}(h_0,t)$ represents the maximum flow velocity between the parallel plates. $u_p(t)$ is the known average inlet velocity and $Q(t)$ is the known inlet volume flow rate function. The above equation is termed additional condition in this paper. The physical configuration of the problem is shown schematically in figure 1.

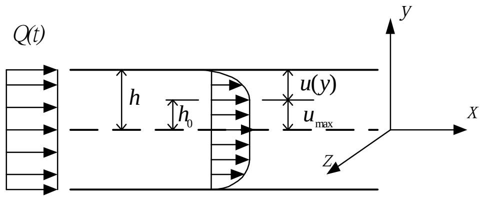

Figure 1 The schematic diagram of the problem

The above governing equations, boundary conditions and initial condition are prescribed and can be solved by the Laplace transform technique, which yields the following equations,

Governing equation

$$\frac{d^2u(y,s)}{dy^2}-\frac{s}{\nu}u(y,s)=\frac{1}{\mu}\frac{dP(x,s)}{dx} \quad (14)$$

Boundary conditions

$$u(h,s)=0 \quad (15)$$

$$\frac{du(h_0,s)}{dy}=0 \quad (16)$$

Additional condition

$$\int_0^{h_0}u_{\max}(h_0,s)dy+\int_{h_0}^{h}u(y,s)dy=u_p(s)h \quad (17)$$

The general solution to the Eq. (14) is

$$u(y,s)=C_1\sinh my+C_2\cosh my+\Psi_p \quad (18)$$

where Ψ_p is the assumed particular solution and

$$m=\sqrt{\frac{s}{\nu}}$$

The boundary conditions (15) and (16) are used to solve for the two arbitrary coefficients C_1 and C_2. Substituting C_1 and C_2 into Eq. (18) gives

$$u(y,s)=\Psi_p\left(1+\frac{\cosh mh_0\cosh my-\sinh mh_0\sinh my}{\sinh mh\sinh mh_0-\cosh mh\cosh mh_0}\right) \quad (19)$$

Let $\Delta=\sinh mh\sinh mh_0-\cosh mh\cosh mh_0$

From the additional condition of Eq. (17), Ψ_p is readily obtained as

$$\Psi_p\left\{\left[1+\frac{(\cosh mh_0)^2-(\sinh mh_0)^2}{\Delta}\right]h_0+\right.$$

$$\left.\int_{h_0}^{h}\left[1+\frac{\cosh mh_0\cosh my-\sinh mh_0\sinh my}{\Delta}\right]dy\right\} \quad (20)$$

$$=u_p(s)2h$$

or

$$\Psi_p=\frac{u_p(s)mh\Delta}{mh\Delta+mh_0\left[(\cosh mh_0)^2-(\sinh mh_0)^2\right]+\Xi} \quad (21)$$

where

$$\Xi=\cosh mh_0\left(\sinh mh-\sinh mh_0\right)$$
$$-\sinh mh_0\left(\cosh mh-\cosh mh_0\right)$$

Substituting Ψ_p into Eq. (19) gives

$$u(y,s)=u_p(s)\cdot\Omega(y,s), \quad (22)$$

where

$$\Omega(y,s)=\frac{mh(\Delta+\cosh mh_0\cosh my-\sinh mh_0\sinh my)}{mh\Delta+mh_0\left[(\cosh mh_0)^2-(\sinh mh_0)^2\right]+\Xi}. \quad (23)$$

Taking the inverse Laplace transform, the velocity profile is

$$u(y,t)=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}u_p(s)\Omega(y,s)e^{st}ds. \quad (24)$$

Furthermore, the pressure gradient is found by substituting Eq. (19) into Eq. (14) to give

$$\frac{dP(x,s)}{dx}=-\rho\, s\Psi_p, \quad (25)$$

or

$$\frac{dP(x,s)}{dx}=-\rho s\frac{u_p(s)mh\Delta}{mh\Delta+mh_0\left[(\cosh mh_0)^2-(\sinh mh_0)^2\right]+\Xi}. \quad (26)$$

Using the inverse transform formula, the pressure gradient distribution also can be obtained.

4. Illustration of examples

Hereafter, we will solve the cases proposed by Das et al. [6] with the Bingham fluid to understand the different flow characteristics between these two fluids under the same condition.

For the first case, the piston velocity u_p (t) moves with a constant acceleration, and the second one, the piston starts suddenly from rest and then maintains this velocity. These two solutions are used to assess the trapezoidal motion of the piston, i.e. the piston has three stages: constant acceleration of piston starting from rest, a period of constant velocity, and a constant deceleration of the piston to a stop. Finally, the oscillatory piston motion is also considered.

4.1 Constant acceleration piston motion

The piston motion of constant acceleration can be described by the following equation.

$$u_p(t)=a_p t=\left(\frac{U_p}{t_0}\right)t, \quad (27)$$

where a_p is the constant acceleration, U_p is the final velocity after acceleration, and t_0 is the time period of acceleration.

Taking the Laplace transform of Eq. (27)

$$u_p(s)=\frac{U_p}{t_0 s^2}. \quad (28)$$

From Eq. (24) and (29), we obtain the velocity profile as

$$u(y,t)=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{U_p}{t_0 s^2}\Omega(y,s)e^{st}ds \quad (29)$$

From the above expression, the integration is determined using complex variable theory, as discussed by. Arpaci [9]. It is easily observed that $s=0$ is a pole of order 2. Therefore, the residue at $s=0$ is

$$\mathrm{Re}\,s(0)=-\frac{U_p h}{t_0}\frac{(\mathrm{X}_1+\mathrm{X}_2+\mathrm{X}_3)}{\left(\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}\right)\left(\frac{1}{\nu}\right)^{\frac{3}{2}}}. \quad (30)$$

where

$$\mathrm{X}_1=\left[\frac{1}{6}(h^3-y^3)h_0+\frac{1}{6}(h-y)h_0^3-\frac{1}{4}(h^2-y^2)h_0^2-\frac{1}{24}(h^4-y^4)\right]\left(\frac{1}{\nu}\right)^{\frac{5}{2}}$$

$$\mathrm{X}_2=\left[(h-y)h_0-\frac{1}{2}(h^2-y^2)\right]t\left(\frac{1}{\nu}\right)^{\frac{3}{2}},$$

$$X_3=\left(\frac{h^5}{30}-\frac{h^4h_0}{8}+\frac{h^3h_0^2}{6}-\frac{h^2h_0^3}{4}+\frac{21h_0^5}{120}\right)\times\left[\frac{\frac{1}{2}\left(h^2-y^2\right)-\left(h-y\right)h_0}{\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}}\right]\left(\frac{1}{\nu}\right)^{\frac{5}{2}}.$$

The other singular points are the roots of following transcendental equation

$$mh\Delta+mh_0\left[\left(\cosh mh_0\right)^2-\left(\sinh mh_0\right)^2\right]+\Xi=0 \quad (31)$$

Setting $m=i\alpha$,

$$\begin{aligned}&\alpha h_0-\alpha h\left(\sin\alpha h\sin\alpha h_0+\cos\alpha h\cos\alpha h_0\right)\\&+\cos\alpha h_0\left(\sin\alpha h-\sin\alpha h_0\right)-\sin\alpha h_0\\&\times\left(\cos\alpha h-\cos\alpha h_0\right)=0\end{aligned} \quad (32)$$

If α_n, $n=1,2,3,\dots\infty$ are zeros of Eq. (32), then $s_n=-\alpha_n^2\nu$, $n=1,2,3,\dots\infty$ are the poles. Since all α_n s are symmetrically placed about zero on the real axis, all the poles (s_n) lie on the negative real axis. These are simple poles, and residues at all these poles can be obtained as

$$\operatorname{Re}s(s_n)=\frac{2U_ph\left[\begin{array}{l}\sin\alpha_nh_0\left(\sin\alpha_nh-\sin\alpha_ny\right)\\+\cos\alpha_nh_0\left(\cos\alpha_nh-\cos\alpha_ny\right)\end{array}\right]e^{-\alpha_n^2\nu t}}{t_0\alpha_n^2\nu\Theta} \quad (33)$$

where

$$\Theta=\left[\begin{array}{l}h_0\left(1-\sin\alpha_nh_0\sin\alpha_nh-\cos\alpha_nh_0\cos\alpha_nh\right)\\-\alpha_nh(h-h_0)\left(\sin\alpha_nh_0\cos\alpha_nh-\sin\alpha_nh\cos\alpha_nh_0\right)\end{array}\right]$$

Adding *Res* (*0*) and *Res* (s_n), a complete solution for constant acceleration case is obtained as

$$\frac{u(y,t)}{U_p}=-\frac{h}{t_0}\frac{\left(X_1+X_2+X_3\right)}{\left(\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}\right)\left(\frac{1}{\nu}\right)^{\frac{3}{2}}}+\frac{2h}{t_0\nu}\sum_{n=1}^{\infty}\frac{\left[\begin{array}{l}\sin\alpha_nh_0\left(\sin\alpha_nh-\sin\alpha_ny\right)\\+\cos\alpha_nh_0\left(\cos\alpha_nh-\cos\alpha_ny\right)\end{array}\right]e^{-\alpha_n^2\nu t}}{\alpha_n^2\Theta} \quad (34)$$

The first term on the right-hand side of Eq. (34) represents the steady state velocity, the second term the transient response of the flow to an abrupt change either in the boundary conditions, body forces, pressure gradient or other external driving force.

The exact solution is compared to Das et al. [6], and it is found when h_0 is equal to zero, these two velocity profiles are identical. It means that Bingham fluid reduces to Newtonian fluid.

Eq. (26) is used to determine the pressure gradient in this flow field, and follows the same procedure for solving velocity profile.

$$\operatorname{Re}s(0)=\frac{\rho U_p}{t_0}\frac{X_4+X_5}{\left(\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}\right)\left(\frac{1}{\nu}\right)^{\frac{3}{2}}}, \quad (35)$$

where

$$X_4=\left(h^2h_0-\frac{1}{2}hh_0^2-\frac{1}{2}h^3\right)\left(\frac{1}{\nu}\right)^{\frac{3}{2}}-ht\left(\frac{1}{\nu}\right)^{\frac{1}{2}},$$

$$X_5=\left(\frac{h^5}{30}-\frac{h^4h_0}{8}+\frac{h^3h_0^2}{6}-\frac{h^2h_0^3}{4}+\frac{21h_0^5}{120}\right)\times\left[\frac{h\nu}{\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}}\right]\left(\frac{1}{\nu}\right)^{\frac{5}{2}}.$$

and the residue at $s=s_n$ is

$$\operatorname{Re}s(s_n)=\frac{2\rho U_ph\left[\begin{array}{l}\sin\alpha_nh\sin\alpha_nh_0\\+\cos\alpha_nh\cos\alpha_nh_0\end{array}\right]e^{-\alpha_n^2\nu t}}{t_0\Theta} \quad (36)$$

Therefore, the pressure gradient is

$$\frac{dp(x,t)}{dx}=\frac{\rho U_p}{t_0}\left[\begin{array}{c}\frac{X_4+X_5}{\left(\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}\right)\left(\frac{1}{\nu}\right)^{\frac{3}{2}}}\\+2h\sum_{n=1}^{\infty}\frac{\left(\begin{array}{l}\sin\alpha_nh\sin\alpha_nh_0\\+\cos\alpha_nh\cos\alpha_nh_0\end{array}\right)e^{-\alpha_n^2\nu t}}{\Theta}\end{array}\right] \quad (37)$$

4.2 Suddenly started flow

For a suddenly started flow between the parallel plates is given as

$$\begin{aligned}u_p&=0, &&\text{for t}\le 0\\&=U_p, &&\text{for t}>0\end{aligned} \quad (38)$$

where U_p is the constant velocity. In which case the velocity is

$$\frac{u(y,t)}{U_p}=\frac{3}{2}\frac{\left[\left(1-\left(\frac{y}{h}\right)^2\right)-2\left(\frac{h_0}{h}\right)\left[1-\left(\frac{y}{h}\right)\right]\right]}{\left[1-\frac{3}{2}\left(\frac{h_0}{h}\right)+\frac{9}{8}\left(\frac{h_0}{h}\right)^3\right]} \tag{39}$$

$$-2h\sum_{n=1}^{\infty}\frac{\begin{bmatrix}\sin\alpha_n h_0\left(\sin\alpha_n h-\sin\alpha_n y\right)\\+\cos\alpha_n h_0\left(\cos\alpha_n h-\cos\alpha_n y\right)\end{bmatrix}e^{-\alpha_n^2\nu t}}{\Theta}$$

and the pressure gradient is

$$\frac{dp(x,t)}{dx}=-\frac{3U_p\mu}{h^2}-2\mu U_p h \times\sum_{n=1}^{\infty}\frac{\alpha_n^2\begin{pmatrix}\sin\alpha_n h\sin\alpha_n h_0\\+\cos\alpha_n h\cos\alpha_n h_0\end{pmatrix}e^{-\alpha_n^2\nu t}}{\Theta}\Bigg] \tag{40}$$

4.3 Trapezoidal piston motion

The prescribed piston velocity is assumed to vary with time as follows:

$$\begin{aligned} u_p(t) &= \frac{U_p}{t_0}t && \text{for } 0\le t\le t_0\\ &= U_p && \text{for } t_0\le t\le t_1\\ &= U_p\frac{(t_2-t)}{(t_2-t_1)} && \text{for } t_1\le t\le t_2\\ &= 0 && \text{for } t_2\le t\le\infty \end{aligned} \tag{41}$$

The solutions to the different time period are as follows:

(i) During the constant acceleration period ($0\le t\le t_0$)

$$\frac{u(y,t)}{U_p}=-\frac{h}{t_0}\frac{\left[X_1+X_2+X_3\right]}{\left(\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}\right)\left(\frac{1}{\nu}\right)^{\frac{3}{2}}}+\frac{2h}{t_0\nu}\sum_{n=1}^{\infty}\frac{\begin{bmatrix}\sin\alpha_n h_0\left(\sin\alpha_n h-\sin\alpha_n y\right)\\+\cos\alpha_n h_0\left(\cos\alpha_n h-\cos\alpha_n y\right)\end{bmatrix}e^{-\alpha_n^2\nu t}}{\alpha_n^2\Theta} \tag{42}$$

(ii) During the constant velocity period ($t_0\le t\le t_1$)

$$\frac{u(y,t)}{U_p}=\frac{\left[\frac{1}{2}\left(h^2-y^2\right)-\left(h-y\right)h_0\right]h}{\left(\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}\right)}+\frac{2h}{t_0\nu}\sum_{n=1}^{\infty}\begin{pmatrix}e^{-\alpha_n^2\nu t}\\-e^{-\alpha_n^2\nu(t-t_0)}\end{pmatrix}\times\frac{\begin{bmatrix}\sin\alpha_n h_0\left(\sin\alpha_n h-\sin\alpha_n y\right)\\+\cos\alpha_n h_0\left(\cos\alpha_n h-\cos\alpha_n y\right)\end{bmatrix}}{\alpha_n^2\Theta} \tag{43}$$

(iii) During the constant deceleration period ($t_1\le t\le t_2$)

$$\begin{aligned}\frac{u(y,t)}{U_p}=&\frac{t_2-t}{t_2-t_1}\frac{\left[\frac{1}{2}\left(h^2-y^2\right)-\left(h-y\right)h_0\right]h}{\left(\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}\right)}\\&+\frac{h\left(X_1+X_3\right)}{\left(t_2-t_1\right)\left(\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}\right)\left(\frac{1}{\nu}\right)^{\frac{3}{2}}}\\&+\frac{2h}{\nu}\sum_{n=1}^{\infty}\left(\frac{e^{-\alpha_n^2\nu t}-e^{-\alpha_n^2\nu(t-t_0)}}{t_0}-\frac{e^{-\alpha_n^2\nu(t-t_1)}}{t_2-t_1}\right)\\&\times\sum_{n=1}^{\infty}\frac{\begin{bmatrix}\sin\alpha_n h_0\left(\sin\alpha_n h-\sin\alpha_n y\right)\\+\cos\alpha_n h_0\left(\cos\alpha_n h-\cos\alpha_n y\right)\end{bmatrix}}{\alpha_n^2\Theta}\end{aligned} \tag{44}$$

(iv) After the piston has stopped ($t_2\le t\le\infty$), $u_p(t)$ is the same as described in Eq. (44)

$$\frac{u(y,t)}{U_p}=\frac{2h}{\nu}\sum_{n=1}^{\infty}\left(\frac{e^{-\alpha_n^2\nu t}-e^{-\alpha_n^2\nu(t-t_0)}}{t_0}-\frac{e^{-\alpha_n^2\nu(t-t_1)}-e^{-\alpha_n^2\nu(t-t_2)}}{t_2-t_1}\right)\times\frac{\left[\sin\alpha_n h_0\left(\sin\alpha_n h-\sin\alpha_n y\right)+\cos\alpha_n h_0\left(\cos\alpha_n h-\cos\alpha_n y\right)\right]}{\alpha_n^2\Theta} \tag{45}$$

In addition, the pressure gradient is also found during these three different stages.

(i) During the constant acceleration period ($0\le t\le t_0$)

$$\frac{dp(x,t)}{dx}=\frac{\rho U_p}{t_0}\left[\frac{X_4+X_5}{\left(\frac{h_0^3}{6}-\frac{h_0h^2}{2}+\frac{h^3}{3}\right)\left(\frac{1}{\nu}\right)^{\frac{3}{2}}}+2h\sum_{n=1}^{\infty}\frac{\begin{pmatrix}\sin\alpha_n h\sin\alpha_n h_0\\+\cos\alpha_n h\cos\alpha_n h_0\end{pmatrix}e^{-\alpha_n^2\nu t}}{\Theta}\right] \tag{46}$$

(ii)During the constant velocity period ($t_0\le t\le t_1$)

$$\frac{dp(x,t)}{dx} = \frac{-h\mu U_p}{\left(\frac{h_0^3}{6} - \frac{h_0 h^2}{2} + \frac{h^3}{3}\right)}$$

$$+\frac{2\rho U_p h}{t_0}\sum_{n=1}^{\infty}\left(e^{-\alpha_n^2 \nu t} - e^{-\alpha_n^2 \nu (t-t_0)}\right) \quad (47)$$

$$\times\frac{\left(\sin\alpha_n h \sin\alpha_n h_0 + \cos\alpha_n h \cos\alpha_n h_0\right)}{\Theta}$$

(iii) During the constant deceleration period ($t_1 \le t \le t_2$)

$$\frac{dp(x,t)}{dx} = \frac{t-t_2}{t_2-t_1}\frac{h\mu U_p}{\left(\frac{h_0^3}{6} - \frac{h_0 h^2}{2} + \frac{h^3}{3}\right)}$$

$$-\frac{\rho U_p}{(t_2-t_1)}\frac{\mathrm{X}_4 + ht\left(\frac{1}{\nu}\right)^{\frac{1}{2}} + \mathrm{X}_5}{\left(\frac{h_0^3}{6} - \frac{h_0 h^2}{2} + \frac{h^3}{3}\right)\left(\frac{1}{\nu}\right)^{\frac{3}{2}}} \quad (54)$$

$$+2\rho U_p h\sum_{n=1}^{\infty}\left(\frac{e^{-\alpha_n^2 \nu t} - e^{-\alpha_n^2 \nu (t-t_0)}}{t_0} - \frac{e^{-\alpha_n^2 \nu (t-t_1)}}{t_2 - t_1}\right) \times\frac{\left(\sin\alpha_n h \sin\alpha_n h_0 + \cos\alpha_n h \cos\alpha_n h_0\right)}{\Theta}$$

(iv) After the piston has stopped ($t_2 \le t \le \infty$)

$$\frac{dp(x,t)}{dx} = 2\rho \mathrm{U_p} h\sum_{n=1}^{\infty}\left(\frac{e^{-\alpha_n^2 \nu t} - e^{-\alpha_n^2 \nu (t-t_0)}}{t_0} - \frac{e^{-\alpha_n^2 \nu (t-t_1)} - e^{-\alpha_n^2 \nu (t-t_2)}}{t_2 - t_1}\right) \quad (55)$$

$$\times\frac{\left(\sin\alpha_n h \sin\alpha_n h_0 + \cos\alpha_n h \cos\alpha_n h_0\right)}{\Theta}$$

4.4 Oscillatory piston motion

The oscillating piston motion starting from rest is considered. The piston motion is described as

$$\begin{aligned} u_p &= 0 && \text{for } t \le 0 \\ &= \mathrm{U_0}\sin(\omega\, t) && \text{for } t > 0 \end{aligned} \quad (56)$$

Taking the Laplace transform of Eq. (56), and we have

$$u_p(s) = \frac{\mathrm{U_0}\omega}{s^2 + \omega^2} \quad (57)$$

Substitute Eq. (57) into Eq. (24) to find the velocity profile. The poles are simple poles at $s = \pm i\omega$ and the roots of $mh\Delta + mh_0\left[(\cosh mh_0)^2 - (\sinh mh_0)^2\right] + \Xi = 0$. The solution to the velocity profile is

$$\frac{u(y,t)}{U_0} = \frac{i}{2}\left[e^{-i\omega t}\Omega(y,-i\omega) - e^{i\omega t}\Omega(y,i\omega)\right]$$

$$+2\omega\,\nu h\sum_{n=1}^{\infty}\left\{\frac{\alpha_n^2}{(\alpha_n^4\nu^2 + \omega^2)} \times \frac{\begin{bmatrix}\sin\alpha_n h_0\left(\sin\alpha_n h - \sin\alpha_n y\right) \\ +\cos\alpha_n h_0\left(\cos\alpha_n h - \cos\alpha_n y\right)\end{bmatrix} e^{-\alpha_n^2 \nu t}}{\Theta}\right\} \quad (58)$$

where $\Omega(y,s)$ is defined by Eq. (23)

And the pressure gradient is obtained as

$$\frac{dp(x,t)}{dx} = -\frac{\rho U_0 \omega}{2}\left[e^{i\omega t}\Gamma(i\omega) + e^{-i\omega t}\Gamma(-i\omega)\right] \quad (59)$$

$$+2\nu^2\rho U_0 \omega h\sum_{n=1}^{\infty}\left[\frac{\alpha_n^4}{(\alpha_n^4\nu^2 + \omega^2)} \times \frac{\begin{pmatrix}\sin\alpha_n h \sin\alpha_n h_0 \\ +\cos\alpha_n h \cos\alpha_n h_0\end{pmatrix} e^{-\alpha_n^2 \nu t}}{\Theta}\right]$$

where

$$\Gamma(s) = \frac{mh\Delta}{\begin{Bmatrix} mh\Delta + mh_0\left[\left(\cosh mh_0\right)^2 - \left(\sinh mh_0\right)^2\right] \\ +\cosh mh_0\left(\sinh mh - \sinh mh_0\right) \\ -\sinh mh_0\left(\cosh mh - \cosh mh_0\right)\end{Bmatrix}}$$

and $m = \sqrt{\frac{s}{\nu}}$

5. Concluding remarks

The analytical solutions of velocity profile and pressure gradient to the different types of piston motion that corresponds to the known inlet volume flow rate are solved by Laplace transform technique. From the solutions obtained, the conclusion can be made, when the yielding position h_0 is equal to zero, the velocity and pressure gradient distribution reduce to that of Newtonian fluid.

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