A semi-analytical solution of bio-heat conduction on the three-layer skin is presented. The performance of the typical heat treatment (heating by laser and cooling by fluid at skin surface) is studied. The transient temperature field and thermal damage of skin are investigated. Effects of several parameters on temperature variation and thermal damage are discussed. The results of the article will be useful for heat therapy in clinics. In addition, the presented result is very consistent to that by the finite element method. The semi-analytical method can be easily applied for solving the general problem of heat conduction in any multilayer structure.

**Keywords:** Bio-heat transfer; skin tissue; semi-analytical solution; thermal damage; laser heating.

**Nomenclature**

- \( c_b \): specific heat of blood.
- \( f(t) \): time-dependent boundary condition.
- \( g \): shifting function.
- \( h \): heat convective coefficient.
- \( k_j \): thermal conductivity of \( j \)th layer.
- \( l_i \): coordinate of different layer.
- \( p \): heat transfer per unit area at skin surface.
- \( q_{\text{met},j} \): metabolic heat generation in the skin of \( j \)th layer.
- \( q_{\text{ext},j} \): heat source due to external heating.
- \( s \): transformed variable of time.
- \( T_f \): temperature of cooling fluid.
1. Introduction

In recent years, the research of bio-heat transfer in living tissues has taken a relevant importance because of the medical interest for applying this fundamental knowledge to the heat treatment. Several literatures have been contributed to the research of bio-heat transfer in living tissues. In heat treatment, laser, photocoagulation, magnetic particles, magnetic fluids have been used to destroy dangerous in living tissues. Accurate prediction of temperature and thermal damage of skin can be useful for improving the effect of treatment.

Skin is an important organ of the human body. It include three layers: epidermis, dermis and subcutaneous fat. The thermal properties of skin vary between different layers. The thermal transmission in skin tissue is a very complex process. It involves multiple phenomenological mechanisms including conduction in tissue, convection between blood and tissues, blood perfusion or advection and diffusion through micro-vascular beds, and metabolic heat generation.

Three kind of mathematical model are usually used to study the bio-heat heat transfer: Pennes model, CV mode, and DPL model. Among three models, the Pennes model was widely used in bio-heat transfer because it’s reasonable and simple. Because the skin is layered structure, the analytical solutions are not available. Traditionally, the numerical approaches such as finite element method, boundary element method and finite difference method were used for solving the bio-heat transfer with layered structure. In general, the analytical approach is only used for one layer structure. The solution methods are reviewed as follows:

Vyas and Rustgi obtained an analytical solution by using Green’s function method based on 1D model to study the laser tissue interaction. Deng and Liu used Green’s function method to solve the governing equations in 3D model. Lin obtained the analytical solutions of bio-heat conduction on one-layer skin tissue in Fourier and non-Fourier models and investigated the case of skin tissue subjected to a harmonic heating using separation of variables. Kengne and Lakhssassi studied the Bio-heat transfer problem for one-dimensional spherical biological tissues using the method of separation of variables. The skin tissue was considered as one layer.
structure for all of the above analytical method. Xu analyzed the temperature, burn damage and thermal stress distributions in the multi-layers skin tissue using the finite difference method (FDM) and finite element method (FEM).\textsuperscript{11} Ng developed 1D finite-difference and 2D finite-element bio-heat transfer based numerical models of the skin, and evaluated the thermal efficacy of cooling treatment.\textsuperscript{24, 25} Ng also solved the bio-heat equation using the boundary element method.\textsuperscript{26} Liu \textit{et al.} employed the Laplace transform method to obtain the numerical solution for multilayer skin.\textsuperscript{27} Kumar studied bio-heat transfer model using the finite element wavelet Galerkin method.\textsuperscript{8}

So far, the analytical methods are often used to solve the problem of single layer bio-heat conduction. The traditional numerical methods are generally used to solve the multi-layers bio-heat conduction. In this study, a semi-analytical method will be developed to obtain the solution of bio-heat conduction on multilayer skin tissue. In order to verify the correctness of the presented method, the comparison of the results of the semi-analytical method and the finite element method will be made. Moreover, the effect of the heat convective coefficient, temperature of cooling fluid and blood perfusion rate on temperature and thermal damage distribution will be discussed.

2. The Mathematical Model of Heat Conduction of the Skin Tissue

The geometric model of skin tissues is established as Fig. 1. The governing equations are

\[ \rho_1 c_1 \frac{\partial T_1}{\partial t} = k_1 \frac{\partial^2 T_1}{\partial x^2} - \omega_{b,1} \rho_b c_b (T_1 - T_a) + (q_{\text{met},1} + q_{\text{ext},1}), \quad -l_1 < x < 0 \]

Fig. 1. Structure and idealized model of human skin.

\[ \text{1a} \]
S.-M. Lin & C.-Y. Li

\[ \frac{\rho c}{2} \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} - \omega_b \rho c_b (T - T_a) + (q_{\text{met},2} + q_{\text{ext},2}), \quad 0 < x < l_2, \]  
(1b)

\[ \frac{\rho c_3}{3} \frac{\partial T}{\partial t} = k_3 \frac{\partial^2 T}{\partial x^2} - \omega_b \rho c_b (T - T_a) + (q_{\text{met},3} + q_{\text{ext},3}), \quad l_2 < x < l_3. \]  
(1c)

The boundary conditions are

At \( x = -l_1 \):

\[ k_1 \frac{\partial T_1(-l_1,t)}{\partial x} - hT_1(-l_1,t) = -hT_f - p(t) = f_1(t). \]  
(2)

At \( x = l_3 \):

\[ T_3(l_3,t) = f_3(t) = T_a. \]  
(3)

The continuity conditions of temperature and heat flux are

At the epidermis–dermis (ED) interface, \( x = 0 \):

\[ T_1(0,t) = T_2(0,t), \]  
(4)

\[ k_1 \frac{\partial T_1(0,t)}{\partial x} = k_2 \frac{\partial T_2(0,t)}{\partial x}. \]  
(5)

At the dermis-subcutaneous fat (DS) interface, \( x = l_2 \):

\[ T_2(l_2,t) = T_3(l_2,t), \]  
(6)

\[ k_2 \frac{\partial T_2(l_2,t)}{\partial x} = k_3 \frac{\partial T_3(l_2,t)}{\partial x}. \]  
(7)

The initial temperature is uniform, \( T_j(x,0) = T_{\text{ina}}, j = 1,2,3. \)

### 3. Solution Method

#### 3.1. Transformation of system

The time variable is divided into infinite sub-domains. In the \( i \)th sub-domain, the time variable is expressed as

\[ (i - 1)\Delta t < t = (i - 1)\Delta t + s < i\Delta t, \quad \text{for} \quad 0 < s < \Delta t. \]  
(8)

The corresponding \( j \)th layer temperature in the \( i \)th time sub-domain is

\[ T_j(x,t) = T_{j,i}(x,s), \quad \frac{\partial T_{j,i}(x,t)}{\partial t} = \frac{\partial T_{j,i}(x,s)}{\partial s}. \]  
(9)

Substituting Eqs. (8) and (9) into Eqs. (1)–(7), the governing equations in the \( i \)th time sub-domain are

\[ \frac{\rho c_i}{\partial s} \frac{\partial T_{i,i}}{\partial s} = k_i \frac{\partial^2 T_{i,i}}{\partial x^2} - \omega_b \rho c_b (T_{i,i} - T_a) + (q_{\text{met},i} + q_{\text{ext},i}), \quad -l_1 < x < 0, \]  
(10a)
\[ \rho_2 c_2 \frac{\partial T_{2,i}}{\partial s} = k_2 \frac{\partial^2 T_{2,i}}{\partial x^2} - \omega_{b,2} \rho_2 c_b (T_{2,i} - T_a) + (q_{\text{met},2} + q_{\text{ext},2}) \quad 0 < x < l_2, \quad (10b) \]

\[ \rho_3 c_3 \frac{\partial T_{3,i}}{\partial s} = k_3 \frac{\partial^2 T_{3,i}}{\partial x^2} - \omega_{b,3} \rho_3 c_b (T_{3,i} - T_a) \]
\[ + (q_{\text{met},3} + q_{\text{ext},3}) \quad l_2 < x < l_3, \quad 0 < s < \Delta t. \quad (10c) \]

The boundary conditions in the \(i\)th time sub-domain are:

At \(x = -l_1\):
\[ k_1 \frac{\partial T_{1,i}(-l_1,t)}{\partial x} - hT_{1,i}(-l_1,t) = -hT_f - p_i(t) = f_{1,i}(t). \quad (11) \]

At \(x = l_3\):
\[ T_{3,i}(l_3,s) = f_{3,i}(s) = T_a. \quad (12) \]

The continuity conditions in the \(i\)th time sub-domain are:

At \(x = 0\):
\[ T_{1,i}(0,s) = T_{2,i}(0,s), \quad (13) \]
\[ k_1 \frac{\partial T_{1,i}(0,s)}{\partial x} = k_2 \frac{\partial T_{2,i}(0,s)}{\partial x}. \quad (14) \]

At \(x = l_2\):
\[ T_{2,i}(l_2,s) = T_{3,i}(l_2,s), \quad (15) \]
\[ k_2 \frac{\partial T_{2,i}(l_2,s)}{\partial x} = k_3 \frac{\partial T_{3,i}(l_2,s)}{\partial x}. \quad (16) \]

### 3.2. Solution of transformed system

If the \(\Delta t\) is smaller enough, the linear variation of temperatures at the interfaces are assumed as

At \(x = 0\):
\[ T_{1,i}(0,s) = f_{12,i}(s) = \begin{cases} T_{\text{ina}} + \nu_{12,i} s & i = 1 \\ T_{1,i-1}(0, \Delta t) + \nu_{12,i} s & i > 1. \end{cases} \quad (17) \]

At \(x = l_2\):
\[ T_{2,i}(l_2,s) = f_{23,i}(s) = \begin{cases} T_{\text{ina}} + \nu_{23,i} s & i = 1 \\ T_{2,i-1}(l_2, \Delta t) + \nu_{23,i} s & i > 1. \end{cases} \quad (18) \]
where the gradient parameters \( \{ \nu_{12,i}, \nu_{23,i} \} \) are to be determined. Based on the assumptions (17) and (18), the transformed governing equation, boundary conditions and initial conditions of each layer are summarized as follows:

For the first layer (epidermis), the transformed governing equation is

\[
\rho_1 c_1 \frac{\partial T_{1,i}}{\partial s} = k_1 \frac{\partial^2 T_{1,i}}{\partial x^2} - \omega_{b,1} \rho_0 c_0 (T_{1,i} - T_a) + (q_{\text{net,1}} + q_{\text{ext,1}}) \quad -l_1 < x < 0, \quad 0 < s < \Delta t.
\]

The transformed boundary conditions are

At \( x = -l_1 \):

\[
k_1 \frac{\partial T_{1,i}(-l_1,s)}{\partial x} - h T_{1,i}(-l_1,s) = -h f_i(s) = f_{1,i}(s).
\]

At \( x = 0 \):

\[
T_{1,i}(0,s) = T_{1,i-1}(0,\Delta t) + \nu_{12,i} s = f_{12,i}(s).
\]

The transformed initial condition is

\[
T_{1,i}(x,0) = T_{1,i-1}(x,\Delta t).
\]

For the second layer (dermis), the transformed governing equation is

\[
\rho_2 c_2 \frac{\partial T_{2,i}}{\partial s} = k_2 \frac{\partial^2 T_{2,i}}{\partial x^2} - \omega_{b,2} \rho_0 c_0 (T_{2,i} - T_a) + (q_{\text{net,2}} + q_{\text{ext,2}}) \quad 0 < x < l_2, \quad 0 < s < \Delta t.
\]

The transformed boundary conditions are

At \( x = 0 \):

\[
T_{2,i}(0,s) = T_{2,i-1}(0,\Delta t) + \nu_{12,i} s = f_{12,i}(s).
\]

At \( x = l_2 \):

\[
T_{2,i}(l_2,s) = T_{2,i-1}(l_2,\Delta t) + \nu_{23,i} s = f_{23,i}(s).
\]

The transformed initial condition is

\[
T_{2,i}(x,0) = T_{2,i-1}(x,\Delta t).
\]

For the third layer (Subcutaneous fat), the transformed governing equation is

\[
\rho_3 c_3 \frac{\partial T_{3,i}}{\partial s} = k_3 \frac{\partial^2 T_{3,i}}{\partial x^2} - \omega_{b,3} \rho_0 c_0 (T_{3,i} - T_a) + (q_{\text{net,3}} + q_{\text{ext,3}}) \quad l_2 < x < l_3, \quad 0 < s < \Delta t.
\]

The transformed boundary conditions are

At \( x = l_2 \):

\[
T_{3,i}(l_2,s) = T_{2,i-1}(l_2,\Delta t) + \nu_{23,i} s = f_{23,i}(s).
\]
The transformed initial condition is
\[ T_{3,i}(l_3, s) = f_{3,i}(s) = T_a. \] (29)

The transformed initial condition is
\[ T_{3,i}(x, 0) = T_{3,i-1}(x, \Delta t). \] (30)

So far, given the gradient parameters \( \nu_{12,i}, \nu_{23,i} \), the temperature of each layer can be independently determined. However, the continuity conditions (14) and (16) must be satisfied. Via the conditions (14) and (16) the accurate parameters \( \nu_{12,i}, \nu_{23,i} \) can be easily calculated in Sec. 3.4.

3.2.1. Solution of the first layer

3.2.1.1. Change of variable

The solution of the first-layer transformed system with nonhomogeneous boundary conditions can be determined by the shifting function method developed by Lin. One assumes
\[ T_{1,i}(x, s) = \nu_{1,i}(x, s) + w_{1,i}(x, s), \quad 0 < s \leq \Delta t, \quad -l_1 \leq x \leq 0, \] (31)

where \( w_{1,i}(x, s) = g_1(x)f_{1,i}(s) + g_2(x)f_{12,i}(s) \), in which \( g_i(\xi), i = 1, 2 \) are the shifting functions and \( \nu(\xi, \tau) \) is the transformed function.

Substituting Eq. (31) into Eqs. (19)–(21), two following subsystems are obtained. The first subsystem is expressed in terms of the transformed variable \( \nu(\xi, \tau) \). The transformed governing equation is
\[ \rho_1 c_1 \frac{\partial \nu_{1,i}}{\partial s} = k_1 \frac{\partial^2 \nu_{1,i}}{\partial x^2} - \alpha_{b,1, \rho_0, c_0} \nu_{1,i} + q_{s1,i}, \] (32a)

where
\[ q_{s1,i} = \alpha_{1,i}(x, s) + \beta_{1,i}(x)\nu_{12,i}, \]
\[ \alpha_{1,i}(x, s) = -\rho_1 c_1 \left( g_1(x) \frac{df_{1,i}}{ds} \right) - \alpha_{b,1, \rho_0, c_0} \left[ g_1(x)f_{1,i} + g_2(x)T_{1,i-1}(0, \Delta t) \right] \] (32b)
\[ \beta_{1,i}(x) = -\rho_1 c_1 g_2(x) - \alpha_{b,1, \rho_0, c_0} g_2(x). \]

The associated boundary conditions are
At \( x = -l_1 \):
\[ k_1 \frac{\partial \nu_{1,i}(-l_1, s)}{\partial s} - h\nu_{1,i}(-l_1, s) = 0. \] (33)

At \( x = 0 \):
\[ \nu_{1,i}(0, s) = 0. \] (34)
The corresponding initial conditions are
\[ \nu_{1,i}(x,0) = T_{1,i-1}(x, \Delta t) - w_{1,i}(x,0). \] (35)

The second subsystem is expressed in terms of the shifting function \( g_i(\xi) \). The transformed governing equation is
\[ \frac{d^2 g_i}{dx^2} = 0, \quad i = 1, 2, \quad -l_1 \leq x \leq 0 \] (36)
and the associated boundary conditions
At \( x = -l_1 \):
\[ k_1 \frac{dg_1(-l_1)}{dx} - hg_1(-l_1) = 1, \quad \text{(37)} \]
\[ k_1 \frac{dg_2(-l_1)}{dx} - hg_2(-l_1) = 0. \quad \text{(38)} \]
At \( x = 0 \):
\[ g_1(0) = 0, \quad \text{(39)} \]
\[ g_2(0) = 1. \quad \text{(40)} \]

The solutions of Eqs. (36)–(40) are
\[ g_1(x) = \frac{x}{k_1 + hl_1}, \quad g_2(x) = 1 + \frac{hx}{k_1 + hl_1}. \quad \text{(41)} \]

3.2.1.2. Solution of the transformed variable
(i) Orthogonality of eigenfunctions

The characteristic equation of Eq. (32) is
\[ \rho_1 c_1 \frac{\partial \nu_{1,i}}{\partial t} = k_1 \frac{\partial^2 \nu_{1,i}}{\partial x^2}. \quad \text{(42)} \]

Considering the separation of variable method, the solution is written as
\[ \nu_{1,i}(x, t) = Y_{1,i}(x) G_{1,i}(t). \quad \text{(43)} \]

Substituting Eq. (43) into Eq. (42), one obtains
\[ \frac{d^2 Y_{1,i}}{dx^2} + \lambda Y_{1,i} = 0, \quad \text{(44)} \]
\[ \frac{dG_{1,i}}{dt} + \lambda \frac{k_1}{\rho_1 c_1} G_{1,i} = 0. \quad \text{(45)} \]
Further, substituting Eq. (43) into Eqs. (33) and (34), the corresponding boundary conditions are

At $x = -l_1$:

$$k_1 \frac{dY_{1,i}(-l_1)}{dx} - hY_{1,i}(-l_1) = 0. \quad (46)$$

At $x = 0$:

$$Y_{1,i}(0) = 0. \quad (47)$$

Based on Eqs. (44), (46) and (47), the eigenfunction is

$$Y_{1,i,n}(x) = \sin \sqrt{\lambda_{1,i,n}}x, \quad (48)$$

where $\tan \sqrt{\lambda_{1,i}}l = \frac{k_1 \sqrt{\lambda_{1,i}}}{h}$. Moreover, the orthogonality of eigenfunctions is easily proved and expressed as

$$\delta_{1,i,m} = \int_{-l_1}^{0} Y_{1,i,m}Y_{1,i,m} dx = \frac{l_1}{2} - \frac{\sin [2l_1 \sqrt{\lambda_{1,i,m}}]}{4 \sqrt{\lambda_{1,i,m}}}. \quad (49)$$

(ii) Mode superposition method

Based on the orthogonality conditions (49), the mode superposition method is used to derive the solution of the transformed system composed of Eqs. (32)–(35). The transformed variable is assumed to be

$$\nu_{1,i}(x, s) = \sum_{n=1}^{\infty} Y_{1,i,n}(x)B_{1,i,n}(s). \quad (50)$$

Substituting Eq. (50) into Eq. (32) and multiplying it by $Y_{1,i,m}$ and integrating it from $-l_1$ to 0, Eq. (32) becomes

$$\rho_1 c_1 \frac{dB_{1,i,m}}{ds} + (k_1 \lambda_{1,i,m} + \tilde{\omega}_{b,1} \rho_0 c_0)B_{1,i,m} = \tilde{q}_{l1,i,m}, \quad (51a)$$

where

$$\tilde{q}_{l1,i,m} = \frac{1}{\rho_1 c_1} \frac{1}{\delta_{1,i,m}} \int_{-l_1}^{0} q_{l1,i}Y_{1,i,m} dx = \tilde{\alpha}_{1,i}(s) + \tilde{\beta}_{1,i} \nu_{12,i}, \quad (51b)$$

$$\tilde{\alpha}_{1,i}(s) = \frac{1}{\rho_1 c_1} \frac{1}{\delta_{1,i,m}} \int_{-l_1}^{0} \alpha_{1,i}(x,s)Y_{1,i,m} dx,$$

$$\tilde{\beta}_{1,i} = \frac{1}{\rho_1 c_1} \frac{1}{\delta_{1,i,m}} \int_{-l_1}^{0} \beta_{1,i}(x)Y_{1,i,m} dx.$$
The solution of Eq. (51) is
\[
B_{1,i,m}(s) = e^{-\alpha_{i} s}B_{1,i,m}(0) + e^{-\alpha_{i} s} \int_{0}^{s} e^{-\alpha_{i} \chi} \hat{q}_{i1,i,m} d\chi = \hat{\alpha}_{1,i}(s) + \hat{\beta}_{1,i}(s) \nu_{12,i}, \tag{52a}
\]
where
\[
\hat{\alpha}_{1,i}(s) = e^{-\alpha_{i} s}B_{1,i,m}(0) + e^{-\alpha_{i} s} \int_{0}^{s} e^{-\alpha_{i} \chi} \hat{\alpha}_{1,i}(\chi) d\chi,
\]
\[
\hat{\beta}_{1,i}(s) = e^{-\alpha_{i} s} \int_{0}^{s} e^{-\alpha_{i} \chi} \hat{\beta}_{1,i}(\chi) d\chi, \quad \alpha_{1} = \frac{k_{1} \lambda_{1,i,m} + \hat{\omega}_{b1,m} c_{p,b}}{\rho_{1} c_{1}}, \tag{52b}
\]
\[
B_{1,i,m}(0) = \left\{ \begin{array}{ll}
\frac{1}{C_{1,m}} \int_{0}^{l_{i}} [T_{i,a} - w_{1,i}(x,0)] Y_{1,i,m} dx & \text{if } i = 1 \\
\frac{1}{C_{2,m}} \int_{0}^{l_{i}} [T_{i,x-1}(0, \Delta t) - w_{1,i}(x,0)] Y_{1,i,m} dx & \text{if } i > 1
\end{array} \right.,
\]
Substituting Eqs. (41), (50) and (52) back into Eq. (31), the temperature of the first layer is expressed as
\[
T_{1,i}(x, s) = T_{1a,i}(x, s) + T_{1b,i}(x, s) \nu_{12,i}, \tag{53a}
\]
where
\[
T_{1a,i}(x, s) = \sum_{m=1}^{\infty} \hat{\alpha}_{1,i}(s) Y_{1,i,m}(x) + [g_{1}(x)f_{1,i}(s) + g_{2}(x)T_{1,i-1}(0, \Delta t)],
\]
\[
T_{1b,i}(x, s) = \sum_{m=1}^{\infty} \hat{\beta}_{1,i}(s) Y_{1,i,m}(x) + g_{2}(x) s, \tag{53b}
\]
Obviously, the temperature parameters \( \{T_{1a,i}(x, s), T_{1b,i}(x, s)\} \) in Eq. (53) have been determined. But the unknown gradient parameter \( \nu_{12,i} \) needs to be determined later.

### 3.3. Solution of the second layer

In the similar way, the temperature of the second layer can be expressed as
\[
T_{2,i}(x, s) = T_{2a,i}(x, s) + T_{2b,i}(x, s) \nu_{12,i} + T_{2c,i}(x, s) \nu_{23,i}, \tag{54a}
\]
where
\[
T_{2a,i}(x, s) = \sum_{m=1}^{\infty} \hat{\alpha}_{2,i}(s) Y_{2,i,m}(x) + \left[ \frac{x}{l_{2}} T_{3,i-1}(l_{2}, \Delta t) + \left( 1 - \frac{x}{l_{2}} \right) T_{2,i-1}(0, \Delta t) \right],
\]
\[
T_{2b,i}(x, s) = \sum_{m=1}^{\infty} \hat{\beta}_{2,i}(s) Y_{2,i,m}(x) + \left( 1 - \frac{x}{l_{2}} \right) s,
\]
\[
T_{2c,i}(x, s) = \sum_{m=1}^{\infty} \hat{\gamma}_{2,i}(s) Y_{2,i,m}(x) + \frac{x}{l_{2}} s;
\]
\[ Y_{2,i,m}(x) = \sin \frac{m\pi x}{l_2}, \quad m = 1, 2, 3, \ldots; \]

\[ B_{2,i,m}(s) = \alpha_{2,i}(s) + \beta_{2,i}(s)\nu_{12,i} + \gamma_{2,i}(s)\nu_{23,i}. \]

\[ \hat{\alpha}_{2,i}(s) = e^{-\alpha_{2,i} s}B_{2,i,m}(0) + e^{-\alpha_{2,i} s}\int_{0}^{s} e^{\alpha_{2,i} \chi} \hat{\alpha}_{2,i}(\chi)d\chi, \]

\[ \hat{\beta}_{2,i}(s) = e^{-\alpha_{2,i} s}\int_{0}^{s} e^{\alpha_{2,i} \chi} \hat{\beta}_{2,i}(\chi)d\chi, \]

\[ \hat{\gamma}_{2,i}(s) = e^{-\alpha_{2,i} s}\int_{0}^{s} e^{\alpha_{2,i} \chi} \hat{\gamma}_{2,i}(\chi)d\chi; \]

\[ \alpha_{2} = \frac{k_{2}(\frac{\pi}{l_{2}})^{2} + \omega_{b,2}\rho_{b}c_{b}}{\rho_{2}c_{2}}; \]

\[ \bar{q}_{2,i}(x) = \alpha_{2,i}(x) + \beta_{2,i}(x)\nu_{12,i} + \gamma_{2,i}(x)\nu_{23,i}, \]

\[ \hat{\alpha}_{2,i}(s) = \frac{1}{\rho_{2}c_{2}} \frac{1}{b_{2,i,m}} \int_{0}^{l_{2}} \alpha_{2,i}(x,s) Y_{2,i,m}dx, \quad \hat{\beta}_{2,i} = \frac{1}{\rho_{2}c_{2}} \frac{1}{b_{2,i,m}} \int_{0}^{l_{2}} \beta_{2,i}(x) Y_{2,i,m}dx, \]

\[ \hat{\gamma}_{2,i} = \frac{1}{\rho_{2}c_{2}} \frac{1}{b_{2,i,m}} \int_{0}^{l_{2}} \gamma_{2,i}(x) Y_{2,i,m}dx; \]

\[ q_{2,i} = \alpha_{2,i}(x) + \beta_{2,i}(x)\nu_{12,i} + \gamma_{2,i}(x)\nu_{23,i}. \]

\[ \alpha_{2,i}(x,s) = -\omega_{b,2}\rho_{b}c_{b}\left[ \frac{x}{l_{2}} T_{3,i-1}(l_{2},\Delta t) + \left( 1 - \frac{x}{l_{2}} \right) T_{2,i-1}(0,\Delta t) \right] \]

\[ + \omega_{b,2}\rho_{b}c_{b} T_{a} + (q_{\text{met,2}} + q_{\text{ext,2}}), \]

\[ \beta_{2,i}(x) = -\rho_{2}c_{2}\frac{x}{l_{2}} - \omega_{b,2}\rho_{b}c_{b}\left( 1 - \frac{x}{l_{2}} \right) s \]

\[ \gamma_{2,i}(x) = -\rho_{2}c_{2}\left( 1 - \frac{x}{l_{2}} \right) - \omega_{b,2}\rho_{b}c_{b}\frac{x}{l_{2}} s \]

\[ \delta_{2,i,m} = \int_{0}^{l_{2}} Y_{2,i,m} Y_{2,i,n}dx = \begin{cases} 0, & \text{for } m \neq n; \\ \frac{l_{2}}{2}, & \text{for } m = n; \end{cases} \]

\[ B_{2,i,m}(0) = \begin{cases} \frac{1}{\delta_{2,i,m}} \int_{0}^{l_{2}} [T_{\text{ina}} - w_{2,i}(x,0)] Y_{2,i,m}dx & i = 1; \\ \frac{1}{\delta_{2,i,m}} \int_{0}^{l_{2}} [T_{2,i-1}(0,\Delta t) - w_{2,i}(x,0)] Y_{2,i,m}dx & i > 1. \end{cases} \]

Obviously, the temperature parameters \( \{T_{2a,i}(x,s), T_{2b,i}(x,s), T_{2c,i}(x,s)\} \) in Eq. (54) have been determined. But the unknown parameter \( \{v_{12,i}, v_{23,i}\} \) will be determined later.
3.4. Solution of the third layer

The solution of the third layer can be expressed as:

\[ T_{3i}(x, s) = T_{3a,i}(x, s) + T_{3b,i}(x, s) \nu_{23i}, \]  

(55a)

where

\[
T_{3a,i}(x, s) = \sum_{m=1}^{\infty} \hat{\alpha}_{3,i}(s) Y_{3,i,m}(x) + \left[ \left( -\frac{1}{l_2 - l_3} x + \frac{l_2}{l_2 - l_3} \right) T_a + \left( -\frac{1}{l_2 + l_3} x + \frac{l_3}{l_2 + l_3} \right) T_{3,i-1}(l_2, \Delta t) \right],
\]

\[
T_{3b,i}(x, s) = \sum_{m=1}^{\infty} \hat{\beta}_{3,i}(s) Y_{3,i,m}(x) + \left( -\frac{1}{l_2 + l_3} x + \frac{l_3}{l_2 + l_3} \right) s;
\]

\[
B_{3,i,m}(s) = \hat{\alpha}_{3,i}(s) + \hat{\beta}_{3,i}(s) \nu_{23,i},
\]

\[
\hat{\alpha}_{3,i}(s) = e^{-\alpha s} B_{2,i,m}(0) + e^{-\alpha s} \int_0^s e^{\alpha \chi} \hat{\alpha}_{3,i}(\chi) d\chi,
\]

\[
\hat{\beta}_{3,i}(s) = e^{-\alpha s} \int_0^s e^{\alpha \chi} \hat{\beta}_{3,i}(\chi) d\chi,
\]

\[
\hat{\alpha}_{3,i}(s) = \frac{1}{\rho_3 c_3} \frac{1}{\delta_{3,i,m}} \int_{l_2}^{l_3} \alpha_{3,i}(x, s) Y_{3,i,m} dx,
\]

\[
\hat{\beta}_{3,i}(s) = \frac{1}{\rho_3 c_3} \frac{1}{\delta_{3,i,m}} \int_{l_2}^{l_3} \beta_{3,i}(x) Y_{3,i,m} dx;
\]

\[
\hat{\alpha}_{3,i}(s) = \alpha_{3,i}(s) + \nu_{23,i},
\]

\[
\hat{\beta}_{3,i}(s) = \beta_{3,i}(s) + \nu_{23,i}.
\]

\[
\alpha_{3,i}(x, s) = -\omega_{b,3} \rho_3 c_6 \left[ \left( -\frac{1}{l_2 - l_3} x + \frac{l_2}{l_2 - l_3} \right) T_a + \left( -\frac{1}{l_2 + l_3} x + \frac{l_3}{l_2 + l_3} \right) T_{3,i-1}(l_2, \Delta t) \right] + \omega_{b,3} \rho_3 c_6 T_a + (q_{\text{net},3} + q_{\text{ext},3}),
\]

\[
\beta_{3,i}(x) = -\rho_3 c_3 \left( -\frac{1}{l_2 - l_3} x + \frac{l_3}{l_2 - l_3} \right) - \omega_{b,3} \rho_3 c_6 \left( -\frac{1}{l_2 + l_3} x + \frac{l_3}{l_2 + l_3} \right) s;
\]

\[
\delta_{3,i,m} = \int_{l_2}^{l_3} Y_{3,i,m} Y_{3,i,n} dx
\]

\[
= \begin{cases} 
0, & \text{for } m \neq n \\
\frac{1}{4} \left(-2l_2 + 2l_3 \pm \frac{\sin[2(l_2 - l_3)\sqrt{\lambda_{3,i,n}}]}{\sqrt{\lambda_{3,i,n}}} \right), & \text{for } m = n
\end{cases}
\]
Semi-Analytical Solution of Bio-Heat Conduction for Multi-Layers Skin

\[ B_{3,i,m}(0) = \begin{cases} \frac{1}{\delta_{3,i,m}} \int_{l_2}^{l_1} [T_{in} - w_{3,i}(x, 0)]Y_{3,i,m} \, dx, & \text{for } i = 1 \\ \frac{1}{\delta_{3,i,m}} \int_{l_2}^{l_1} [T_{3,i-1}(0, \Delta t) - w_{3,i}(x, 0)]Y_{3,i,m} \, dx, & \text{for } i > 1 \end{cases} \]

\[ Y_{3,i,m}(x) = \cos(\sqrt{\lambda_{3,i,m} x}) \sin(\sqrt{\lambda_{3,i,m} l_2}) - \cos(\sqrt{\lambda_{3,i,m} l_2}) \sin(\sqrt{\lambda_{3,i,m} x}), \]

(55b)

in which \( \cos[\sqrt{\lambda_{3,i,l_3}} - \cos[\sqrt{\lambda_{3,i,l_2}} \sin[\sqrt{\lambda_{3,i,l_3}}] = 0 \).

Obviously, the temperature parameters \( \{T_{3a,i}(x, s), T_{3b,i}(x, s)\} \) in Eq. (55) have been determined. But the unknown parameter \( \nu_{23,i} \) will be determined later.

### 3.5. Determination of gradient parameters

Considering the heat-flux continuity conditions (5) and (7) at \( s = \Delta t \), Eqs. (5) and (7) become

\[ k_1 \frac{\partial T_{1,i}(0, \Delta t)}{\partial x} = k_2 \frac{\partial T_{2,i}(0, \Delta t)}{\partial x}, \]

(56)

\[ k_2 \frac{\partial T_{2,i}(l_2, \Delta t)}{\partial x} = k_3 \frac{\partial T_{3,i}(l_2, \Delta t)}{\partial x}. \]

(57)

Substituting Eqs. (53)–(55) into Eqs. (56) and (57), the gradient parameters of temperature at the interfaces are obtained

\[ \nu_{12,i} = \frac{c_2 c_6 - c_3 c_5}{c_2 c_4 - c_1 c_5}, \quad \nu_{23,i} = \frac{c_3}{c_2} - \frac{c_1}{c_2} \nu_{12,i}, \]

(58a)

where

\[ c_1 = k_2 \frac{\partial T_{2a,i}(0, \Delta t)}{\partial x} - k_2 \frac{\partial T_{2b,i}(0, \Delta t)}{\partial x}, \quad c_2 = k_2 \frac{\partial T_{2c,i}(0, \Delta t)}{\partial x}, \]

\[ c_3 = k_1 \frac{\partial T_{1a,i}(0, \Delta t)}{\partial x} - k_2 \frac{\partial T_{2a,i}(0, \Delta t)}{\partial x}, \quad c_4 = k_2 \frac{\partial T_{2b,i}(l_2, \Delta t)}{\partial x}, \]

\[ c_5 = k_2 \frac{\partial T_{2c,i}(l_2, \Delta t)}{\partial x} - k_3 \frac{\partial T_{3a,i}(l_2, \Delta t)}{\partial x}, \quad c_6 = k_3 \frac{\partial T_{3a,i}(l_2, \Delta t)}{\partial x} - k_2 \frac{\partial T_{2b,i}(l_2, \Delta t)}{\partial x}. \]

(58b)

Substituting Eq. (58) back into Eqs. (17) and (18), the temperature variation at the interfaces in the \( i \)th time sub-domain is obtained. At the meanwhile, if Eq. (58) is substituted back into Eqs. (53)–(55), the temperature distribution in the \( i \)th time sub-domain is also obtained. Obviously, if the time increment \( \Delta t \) approaches to zero, the accurate solution can be determined. Therefore, the overall temperature variation of the multilayer skin can be determined step by step.
4. Thermal Damage

Accurate prediction of thermal damage for skin tissue is helpful for heat therapy. In order to quantify the thermal damage, one can use the method developed by Henriques and Moritz. It can be expressed as follows:

\[ \Omega(t) = \int_0^1 A \exp\left(-E_a/R \right) dt, \tag{59} \]

where \(\Omega\) is the thermal damage index, \(A\) is the frequency factor, \(R\) is the universal gas constant and \(E_a\) is the activation energy. Duck indicated that: \(\Omega = 0.53\), the first degree burn; \(\Omega = 1.0\), the second degree burn; \(\Omega = 10^4\), the third degree burn \((E_a/R = 75000, A = 3.1 \times 10^9)\). For the model given by Henriques and Moritz, the frequency factor \(A\) and the activation energy \(E_a\) are independent to the temperature. Based on Eq. (59), some other models given by Fugitt, Stoll and Greene, Takata and Wu were developed by empirically fitting to experimental data. Their parameters \(\{A, E_a\}\) depend on the temperature. Ng and Chua made the parametric and sensitivity analysis of these models.

5. Numerical Result and Discussion

The following heat conduction of three-layer skin is investigated. Assume that the initial temperature of skin is \(T(x, 0) = T_{in\_a} = 37^\circ C\). At \(t = 0\), the skin surface at \(x = -l_1\), start to be heated under the constant heat flux. At \(t = 15\) s, the heat source is removed. Simultaneously, the skin surface is continuously cooled by fluid from \(t = 0\) to \(t = 90\) s, as shown in Fig. 2. At the inner side of skin, \(T(l_3, t) = 37^\circ C\). The properties of the skin are presented in Table 1.

![Fig. 2. Laser heating and fluid cooling history at the skin surface.](image-url)
Figure 3 shows the effect of the heat convective coefficient $h$ and temperature of cooling fluid $T_f$ on the temperature at the skin surface. The temperature increases with the heating time. If the heat source is removed from the skin surface, the temperature will be abruptly decreased. Finally, if the cooling time is longer enough, the temperature becomes constant. Moreover, it is found that if the heat convective coefficient $h = 10 \text{ w/m}^2\text{C}$, the influence of temperature of cooling fluid $T_f$ on the skin temperature is negligible. If $h = 100 \text{ w/m}^2\text{C}$, the highest temperature with $T_f = 0\text{C}$ is lower by about 4$\text{C}$ than that with $T_f = 20\text{C}$. If $h = 1000 \text{ w/m}^2\text{C}$, the temperature with $T_f = 0\text{C}$ is lower by about 15$\text{C}$ than those with $T_f = 20\text{C}$. It is concluded that the higher the heat convective coefficient $h$ is, the greater the effect of the cooling temperature $T_f$ on skin temperature is.

Figure 4 demonstrates the effect of $h$ and $T_f$ on the temperature variation at the ED interface. Their effects are similar to those at the skin surface shown in Fig. 3.
Figure 5 demonstrates the effect of $h$ and $T_f$ on the temperature variation at the DS interface. Due to the long distance to the heating surface, the variation rate of temperature due to surface heating is significantly lower than those at the skin surface and the ED interface.

Figures 6 and 8 demonstrate effect of the heat convective coefficient $h$ and temperature of cooling fluid $T_f$ on the temperature distribution over the skin at the end of laser heating, $t = 15$ s and at the long cooling time, $t = 90$ s, respectively. It is

Fig. 5. Effect of the heat convective coefficient $h$ and temperature of cooling fluid $T_f$ on the temperature history at interface of dermis and subcutaneous fat ($\omega_{b,1} = 0$, $\omega_{b,2} = 0.1$, $\omega_{b,3} = 0$, $x = 0.15$ mm).
observed from Fig. 6 that at the end of laser heating the temperature decreases with skin depth because of the distance to the heating surface. Moreover, there is no temperature variation in the deeper skin \((x > 3.9 \text{ mm})\) because the heating time \((t_{\text{heating}} = 15 \text{ s})\) is too short for the heat flux \(p = 3 \text{ w/cm}^2\). Moreover, it is found from Fig. 7 that the presented results are very consistent to that by the finite element method. Figure 8 demonstrates that the temperature at the inner side of skin, the temperature \(T(l_3, t) = 37^\circ\text{C}\). Moreover, the shorter the distance from the cooling
surface is, the greater the effects of the heat convection coefficient $h$ and cooling temperature $T_f$.

Figure 9 demonstrates the effects of the heat convective coefficient $h$ and temperature of cooling fluid $T_f$ on the thermal damage index $\Omega$ history at the skin surface. It is found that if $h = 10 \text{ w/m}^2\text{C}$, the difference between the thermal damage index $\Omega$ with $T_f = 0^\circ\text{C}$ and $T_f = 20^\circ\text{C}$ is slight. If $h = 100 \text{ w/m}^2\text{C}$, the thermal damage index $\Omega$ with $T_f = 0^\circ\text{C}$ is lower by about one orders of magnitude.
than those of $T_f = 25\degree C$. Especially if $h = 1000 \text{ w/m}^2\text{C}$, the thermal damage index with $T_f = 0\degree C$ is lower by about four orders of magnitude than those with $T_f = 25\degree C$. If $T_f = 25\degree C$, the thermal damage index with $h = 100 \text{ w/m}^2\text{C}$ is lower by about four orders of magnitude than those with $h = 10 \text{ w/m}^2\text{C}$. It is concluded that increasing the heat convective coefficient $h$ and decreasing the temperature of cooling fluid $T_f$ will significantly decrease the skin temperature and the thermal damage.

![Graph showing the effect of heat convective coefficient $h$ and temperature of cooling fluid $T_f$ on thermal damage](image1.png)

**Fig. 10.** Effect of the heat convective coefficient $h$ and temperature of cooling fluid $T_f$ on the thermal damage $\Omega$ distribution along skin depth at $t = 90s$ ($\omega_{b,1} = 0$, $\omega_{b,2} = 0.1$, $\omega_{b,3} = 0$).

![Graph showing the effect of blood perfusion rate $\omega_{b,2}$ on temperature history](image2.png)

**Fig. 11.** Effect of the blood perfusion rate $\omega_{b,2}$ on the temperature history at skin surface ($\omega_{b,1} = 0$, $\omega_{b,3} = 0$, $x = -0.1 \text{ mm}$, $h = 100$, $T_f = 20\degree C$).
Figure 10 shows effects of the heat convective coefficient \( h \) and temperature of cooling fluid \( T_f \) on the thermal damage distribution of skin at \( t = 90 \) s. It is found that if \( h = 1000 \text{ w/m}^2\text{C} \), the thermal damage is slight. If \( h = 100 \text{ w/m}^2\text{C} \) or \( h = 10 \text{ w/m}^2\text{C} \), the thermal damage reach the third-degree burns (\( \Omega > 10^4 \)) at the skin surface and the ED interface. At the DS interface the thermal damage is slight (\( \Omega < 0.25 \)).

Figure 12. Effect of the blood perfusion rate \( \varpi_{b,2} \) on the temperature history at interface of epidermis and dermis (\( \varpi_{b,1} = 0, \varpi_{b,3} = 0, x = 0, h = 100, T_f = 20^\circ\text{C} \)).

Figure 13. Effect of the blood perfusion rate \( \varpi_{b,2} \) on the temperature history at interface of dermis and subcutaneous fat (\( \varpi_{b,1} = 0, \varpi_{b,3} = 0, x = 0, h = 100, T_f = 20^\circ\text{C} \)).
Figures 11 and 12 show the effect of the blood perfusion rate $b_2$ on the temperature history at the skin surface and at the ED interface. It is found that the smaller the blood perfusion rate $b_2$, the faster temperature increasing due to the surface heating and the higher the temperature. In other words, the attribute of the blood perfusion is the damping to the heat impact. The fact is obviously verified in Fig. 13 which demonstrates the effect of blood perfusion rate $b_2$ on the temperature history at the DS interface.

Fig. 14. Effect of the blood perfusion rate $b_2$ on the temperature distribution along skin depth at $t = 15s$ ($b_{0,1} = 0$, $b_{0,3} = 0$, $x = 0$, $h = 100$, $T_f = 20^\circ C$).

Fig. 15. Effect of the blood perfusion rate $b_2$ on the temperature distribution along skin depth at different time ($b_{0,1} = 0$, $b_{0,3} = 0$, $x = 0$, $h = 100$, $T_f = 20^\circ C$).
Figure 14 demonstrates the effect of the blood perfusion rate on the temperature distribution over the skin at the end of laser heating, \( t = 15 \) s, Fig. 15 from \( t = 90 \) s to \( t = 1000 \) s. It is found from Fig. 14 that the smaller the blood perfusion rate \( \omega_{b,2} \) is, the higher the temperature is. Figure 15 shows that if \( \omega_{b,2} = 0 \), the difference between the temperature distributions with \( t = 90 \) s and \( t = 500 \) s is significant. It is because for \( \omega_{b,2} = 0 \) there is no damping to the heat impact. Moreover, if the cooling time is over critical period, the temperature becomes in the steady state.

Figure 16. Effect of the blood perfusion rate \( \omega_{b,2} \) on the thermal damage \( \Omega \) history at skin surface (\( \omega_{b,1} = 0, \omega_{b,2} = 0.1, \omega_{b,3} = 0, x = -0.1 \) mm).

Figure 17. Effect of the blood perfusion rate \( \omega_{b,2} \) on the thermal damage \( \Omega \) distribution along skin depth at \( t = 90 \) s (\( \omega_{b,1} = 0, \omega_{b,2} = 0.1, \omega_{b,3} = 0 \)).
smaller blood perfusion rate is, the longer the critical period to the steady state is. In addition, the smaller blood perfusion rate $b_{2}$, the lower the steady temperature is.

Figure 16 shows effect of the blood perfusion rate $b_{2}$ on the thermal damage $\Omega$ history at the skin surface. It is found that the smaller the blood perfusion rate is, the greater the thermal damage is. Figure 17 shows the effect of the blood perfusion rate $b_{2}$ on the thermal damage distribution over the skin at $t = 90$ s. The thermal damage will decrease significantly with the distance from the heating surface.

6. Conclusion

In this paper, the semi-analytical solution of bio-heat conduction on three-layer skin tissue is presented. The effects of different parameters on the temperature and thermal damage have been investigated. The main phenomena are revealed as follow:

(a) Increasing the heat convective coefficient $h$ and decreasing the temperature of cooling fluid $T_f$ will significantly decrease the skin temperature and the thermal damage.

(b) The higher the heat convective coefficient $h$ is, the greater the effect of the cooling temperature $T_f$ on skin temperature is.

(c) The attribute of the blood perfusion is the damping to the heat impact. The larger the blood perfusion rate is, the weaker the thermal damage is.

(d) The smaller blood perfusion rate is, the longer the critical period to the steady state is.

(e) The smaller blood perfusion rate $b_{2}$, the lower the steady temperature is.

Acknowledgment

The support of the Ministry of Science and Technology of Taiwan, R. O. C., is gratefully acknowledged (Grant number: MOST 103-2622-E-168-015-CC3).

References


29. Dahan S, Lagarde JM, Turlier V, Courrech L, Mordon S, Treatment of neck lines and forehead rhytids with a nonablative 1540 nm Er: Glass laser: A controlled clinical study


