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ROBUST STABILITY OF DISCRETE UNCERTAIN TIME-DELAY SYSTEMS BY USING A SOLUTION BOUND OF THE LYAPUNOV EQUATION

Chien-Hua Lee[1], Tsung-Lieh Hsien[2] and Cheng-Yi Chen[1]

[1]Department of Electrical Engineering
Cheng-Shiu University
No. 840, Chengcing Rd., Niaosong Township, Kaohsiung 833, Taiwan
chienhua@csu.edu.tw

[2]Department of Electrical Engineering
Kung Shan University
No. 949, Da Wan Rd., Yung Kang City, Tainan 710, Taiwan
tlhsien@mail.ksu.edu.tw

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Abstract. *This paper addresses the robust stability test problem for discrete time-delay systems subjected to uncertainties. By using Lyapunov stability theorem associated with solution bound of Lyapunov equation, a concise criterion is presented to guarantee the robust stability of the time-delay systems. Then, this approach is applied to discuss the same problem for time-delay systems with nonlinear uncertainties. A robust stability condition is also obtained. The feature of these present criteria is that they are independent of any Lyapunov equation although the Lyapunov approach is adopted.*
Keywords: Solution bound, Robust stability, Time-delay, Uncertainty

1. **Introduction.** For controlled systems, stability is one of the important characteristics and the most fundamental requirement. In practice, due to the information transmission, computation of variables, etc., time delay(s) exist(s) in real-life systems [5,6]. Besides, uncertainties that occurred as a result of using approximate system model for simplicity, data errors for evaluation, aging, etc., also exist naturally in control systems. Therefore, both of time delay(s) and uncertainties should be considered in system model. It is known that time delay(s) and uncertainties will increase the numbers of eigenvalue or cause the variations of eigenvalues of linear systems and hence are always sources of instability of controlled systems. Therefore, the research of the impact of time delay(s) and/or uncertainties for linear or nonlinear systems has become more and more attractive during the past decades. In recent years, a number of research results have been devoted to stability analysis and/or stabilization controller design for uncertain time-delay systems [1-4,7-9]. The objective of this paper is to derive simple robust stability conditions for discrete uncertain time-delay systems. By using the Lyapunov equation approach associated with an upper solution bound of the discrete Lyapunov equation, a new stability criterion is derived for the time-delay systems. It is seen that it is tighter than a well-known result proposed in [9]. According to this new approach, we then solve the same problem for time-delay systems with nonlinear uncertainties and establish a concise stability condition. An interesting consequence is that these obtained criteria do not involve any Lyapunov equation. Finally, we also give numerical examples to demonstrate the merits and verify the correctness of the presented schemes.

The following symbol conventions are used in this paper. Symbols $\mathbb{R}$, A^T, $\lambda_1(A)$, $x^T(t)$, $\|x(t)\|$ and $\|A\|$, respectively, means real number field, transpose of matrix A, the maximal eigenvalue of a symmetric matrix A, transpose of vector $x(t)$, norm of vector $x(t)$ with $\|x(t)\| = \left(x^T(t)x(t)\right)^{1/2}$, and induced norm of matrix A with $\|A\| = \lambda_1(A^TA)^{1/2}$.

2. **Main Results.**

Consider the discrete time-delay system

$$x(k+1) = Ax(k) + Bx(k-d) \tag{1}$$

where $x(\cdot) \in \mathbb{R}^n$ represents the state, integer $d > 0$ denotes the delay, A and B are constant matrices with appropriate dimensions and A is a stable matrix.

Before deriving the robust stability criteria, we first review the following result.

Lemma 2.1. [10] *Matrices A, $B \in \mathbb{R}^{n\times n}$. Then for any given positive constant α, the following inequality is satisfied.*

$$A^TB + B^TA \le \alpha A^TA + \frac{1}{\alpha}B^TB. \tag{2}$$

Utilizing the Lyapunov equation approach and Lemma 2.1, we derive the following criteria.

Theorem 2.1. *The time-delay system (1) is robustly stable if the following condition holds.*

$$(\|A\| + \|B\|)\left(\frac{A^TA}{\|A\|} + \frac{B^TB}{\|B\|}\right) < I. \tag{3}$$

Proof: For system (1), a Lyapunov function is constructed as

$$V(x(k)) = x^T(k)Px(k) + \frac{\|A\| + \|B\|}{\|B\|}\sum_{j=1}^{d} x^T(k-j)B^TPBx(k-j) \tag{4}$$

where the positive definite matrix P satisfies the following Lyapunov equation.

$$A^TPA - P = -Q. \tag{5}$$

Here, the positive matrix Q is selected as

$$Q = q\left(\frac{\|B\|}{\|A\|}A^TA + \frac{\|A\| + \|B\|}{\|B\|}B^TB + \varepsilon I\right) \tag{6}$$

where q is an arbitrary positive constant and $\varepsilon \to 0$ is a positive constant.

We rewrite the Lyapunov Equation (5) as

$$A^T(qI-P)A - (qI-P) = Q + qA^TA - qI = q\left[(\|A\| + \|B\|)\left(\frac{A^TA}{\|A\|} + \frac{B^TB}{\|B\|}\right) + \varepsilon I - I\right]. \tag{7}$$

It is seen that if the condition (3) is satisfied then there must exist $\varepsilon \to 0$ such that the right-hand side of (7) is a negative definite matrix. This means that the Lyapunov Equation (7) has a positive solution and we have

$$P < qI. \tag{8}$$

For convenience, we use V, x and x_d to replace $V(x(k))$, $x(k)$ and $x(k-d)$, respectively, in the following and later descriptions. Taking the forward difference for the Lyapunov function (4) yields.

$$\begin{aligned}\Delta V &= x^T(k+1)Px(k+1) - x^TPx + \frac{\|A\| + \|B\|}{\|B\|}\left(x^TB^TPBx - x_d^TB^TPBx_d\right)\\ &= x^T\left(A^TPA - P\right)x + x^TA^TPBx_d + x_d^TB^TPAx + x_d^TB^TPBx_d\\ &\quad + \frac{\|A\| + \|B\|}{\|B\|}\left(x^TB^TPBx - x_d^TB^TPBx_d\right).\end{aligned} \tag{9}$$

Furthermore, we have

$$x^T A^T PBx_d + x_d^T B^T PAx \le \frac{\|B\|}{\|A\|} x^T A^T PAx + \frac{\|A\|}{\|B\|} x_d^T B^T PBx_d. \tag{10}$$

Substituting the above inequality into (9) leads to

$$\begin{aligned} \Delta V \le & x^T \left(A^T PA - P + \frac{\|B\|}{\|A\|} A^T PA + \frac{\|A\| + \|B\|}{\|B\|} B^T PB \right) x \\ < & x^T \left[-Q + q \left(\frac{\|B\|}{\|A\|} A^T A + \frac{\|A\| + \|B\|}{\|B\|} B^T B \right) \right] x = -qx^T(\varepsilon I)x < 0 \end{aligned} \tag{11}$$

where (6) is used. Thus, it is obvious that if the condition (3) holds, then $\Delta V < 0$ and the time-delay system (1) is robustly stable. Therefore, this completes the proof.

Remark 2.1. *A stability criterion for time-delay system (1) has been given in [9]. We rewrite it as follows.*

$$\|A\| + \|B\| < 1. \tag{12}$$

Obviously, it is very concise. The condition (3) is also concise. Furthermore, we have

$$\frac{A^T A}{\|A\|} + \frac{B^T B}{\|B\|} \le \lambda_1 \left(\frac{A^T A}{\|A\|} + \frac{B^T B}{\|B\|} \right) I \le \lambda_1 \left(\frac{A^T A}{\|A\|} \right) I + \lambda_1 \left(\frac{B^T B}{\|B\|} \right) I = (\|A\| + \|B\|)I. \tag{13}$$

This means that the condition (3) is better than (13).

An interesting consequence of this theorem is that the condition (3) is independent of the constant q and the matrix P although the Lyapunov Equation (5) is used.

Now, we apply the above approach to solve the same problem for uncertain time-delay system. Consider the following discrete uncertain time-delay system

$$x(k+1) = Ax(k) + Bx(k-d) + f(x(k), k) + f_1(x(k-d), k) \tag{14}$$

where $f(x(k), k)$ and $f_1(x(k-d), k)$ are nonlinear uncertainties with the following properties:

$$\|f(x(k), k)\| \le \eta \|x(k)\| \quad \text{and} \quad \|f_1(x(k-d), k)\| \le \gamma \|x(k-d)\|. \tag{15}$$

where η and γ are positive constants.

Theorem 2.2. *If the following condition is met*

$$(\|A\| + \|B\| + \eta + \gamma) \left(\frac{A^T A}{\|A\|} + \frac{B^T B}{\|B\|} + \eta I + \gamma I \right) < I \tag{16}$$

then, the uncertain time-delay system (14) is robustly stable.

Proof: For this case, we choose a Lyapunov function as

$$V(x(k)) = x^T(k)Px(k) + \sum_{j=1}^{d} (\|A\| + \|B\| + \eta + \gamma) x^T(k-j) \left(\frac{B^T PB}{\|B\|} + q\gamma I \right) x(k-j) \tag{17}$$

where the positive definite matrix P satisfies the Lyapunov Equation (5) with

$$Q = q \left[(\|B\| + \eta + \gamma) \frac{A^T A}{\|A\|} + (\|A\| + \|B\| + \eta + \gamma) \left(\frac{B^T B}{\|B\|} + \eta I + \gamma I \right) \right] \tag{18}$$

where q is an arbitrary positive constant. We rewrite the Lyapunov Equation (5) as

$$A^T(qI - P)A - (qI - P)$$

$$
\begin{aligned}
&= q\left[(\|B\|+\eta+\gamma)\frac{A^TA}{\|A\|}+(\|A\|+\|B\|+\eta+\gamma)\left(\frac{B^TB}{\|B\|}+\eta I+\gamma I\right)+A^TA-I\right] \\
&= q\left[(\|A\|+\|B\|+\eta+\gamma)\left(\frac{A^TA}{\|A\|}+\frac{B^TB}{\|B\|}+\eta I+\gamma I\right)-I\right].
\end{aligned} \tag{19}
$$

It is seen that the condition (16) can assure the solution $(qI - P)$ of Equation (19) is positive definite. That is, $P < qI$. In the following and later descriptions, we use f and f_1 to replace $f(x(k), k)$ and $f_1(x(k-d), k)$, respectively, for convenience. Taking the forward difference for $V(x(k))$ along the trajectories of system (14) results in

$$
\begin{aligned}
\Delta V = \; & x^T\left[A^TPA-P\right]x+x^TA^TPBx_d+x^TA^TPf+x^TA^TPf_1+x_d^TB^TPAx \\
&+f^TPAx+f_1^TPAx+x_d^TB^TPBx_d+x_d^TB^TPf+x_d^TB^TPf_1+f^TPBx_d \\
&+f^TPf+f^TPf_1+f_1^TPBx_d+f_1^TPf+f_1^TPf_1+(\|A\|+\|B\|+\eta \\
&+\gamma)\left[x^T\left(\frac{B^TPB}{\|B\|}+q\gamma I\right)x-x_d^T\left(\frac{B^TPB}{\|B\|}+q\gamma I\right)x_d\right].
\end{aligned} \tag{20}
$$

Using (10), $P < qI$ and the facts

$$
\begin{aligned}
x^TA^TPf+f^TPAx &\le \frac{\eta}{\|A\|}x^TA^TPAx+\frac{\|A\|}{\eta}f^TPf<\frac{\eta}{\|A\|}x^TA^TPAx+\frac{q\|A\|}{\eta}\eta^2\|x\|^2 \\
&= x^T\left(\frac{\eta}{\|A\|}A^TPA+q\eta\|A\|I\right)x,
\end{aligned} \tag{21}
$$

$$
x^TA^TPf_1+f_1^TPAx \le \frac{\gamma}{\|A\|}x^TA^TPAx+\frac{\|A\|}{\gamma}f_1^TPf_1<\frac{\gamma}{\|A\|}x^TA^TPAx+x_d^Tq\gamma\|A\|Ix_d, \tag{22}
$$

$$
x_d^TB^TPf+f^TPBx_d \le \frac{\eta}{\|B\|}x_d^TB^TPBx_d+\frac{\|B\|}{\eta}f^TPf<\frac{\eta}{\|B\|}x_d^TB^TPBx_d+x^Tq\eta\|B\|Ix, \tag{23}
$$

$$
x_d^TB^TPf_1+f_1^TPBx_d \le \frac{\gamma}{\|B\|}x_d^TB^TPBx_d+\frac{\|B\|}{\gamma}f_1^TPf_1<x_d^T\left[\frac{\gamma}{\|B\|}B^TPB+q\gamma\|B\|I\right]x_d, \tag{24}
$$

$$
f^TPf_1+f_1^TPf \le \frac{\gamma}{\eta}f^TPf+\frac{\eta}{\gamma}f_1^TPf_1<q\gamma\eta\left[x^Tx+x_d^Tx_d\right]. \tag{25}
$$

Equation (20) becomes

$$
\begin{aligned}
\Delta V <\; & x^T\left[-Q+\frac{\|B\|+\eta+\gamma}{\|A\|}A^TPA+q\eta(\|A\|+\|B\|+\eta+\gamma)I\right]x \\
&+(\|A\|+\|B\|+\eta+\gamma)x_d^T\left(B^TPB\|B\|^{-1}+q\gamma I\right)x_d+(\|A\|+\|B\| \\
&+\eta+\gamma)\left[x^T\left(\frac{B^TPB}{\|B\|}+q\gamma I\right)x-x_d^T\left(\frac{B^TPB}{\|B\|}+q\gamma I\right)x_d\right] \\
<\; & x^T\left[-Q+q\left[\frac{\|B\|+\eta+\gamma}{\|A\|}A^TA+(\|A\|+\|B\|+\eta\right.\right. \\
&\left.\left.+\gamma)\left(\frac{B^TB}{\|B\|}+\eta I+\gamma I\right)\right]\right]x=0.
\end{aligned} \tag{26}
$$

It shows if the condition (16) holds then ΔV is negative which guarantees the robust stability of system (14). Thus, the proof is completed.

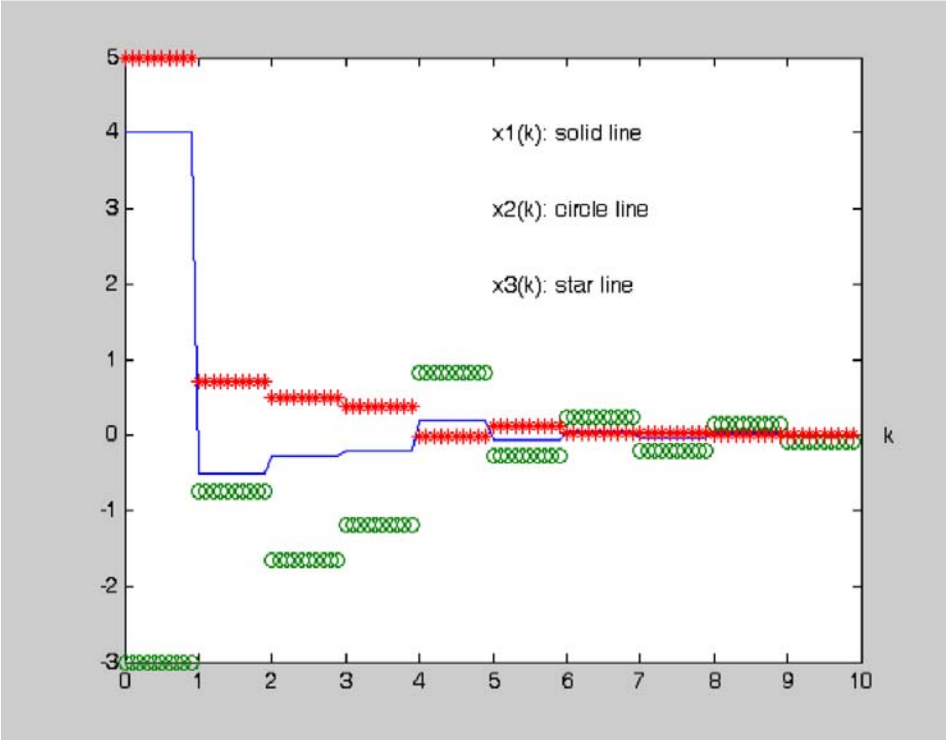

FIGURE 1. The trajectories of state vector $x(k)$ of Example 3.1

3. **Numerical Examples.** To demonstrate the applicability of the presented schemes, we give the following examples.

Example 3.1. *Consider the discrete time-delay system (1) with*

$$A = \begin{bmatrix} -0.1 & -0.1 & 0 \\ 0 & -0.5 & -0.05 \\ 0 & 0.1 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & 0 & 0 \\ -0.5 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

By the simple condition (12), we have $\|A\| + \|B\| = 1.0343 > 1$*. This condition can not judge the stability for this case. We now check the condition of Theorem 2.1 and obtain*

$$(\|A\| + \|B\|)\, \lambda_1 \left(\frac{A^T A}{\|A\|} + \frac{B^T B}{\|B\|} \right) = 0.5643 < 1.$$

Our result can assure that this system is robustly stable. Let $d = 2$ *and* $x(k) = [4\ -3\ 5]^T$ *for* $k \in [-2,\ 0]$*. The simulation results for the states are shown in Figure 1. It is seen that all states are regulated to zero irrespective of the time-delay and uncertainties.*

Example 3.2. *Consider the uncertain time-delay system (14) where matrices A and B are the same as those in Example 3.1. From (16), we can obtain*

$$(\|A\| + \|B\| + \eta + \gamma) \left[\lambda_1 \left(\frac{A^T A}{\|A\|} + \frac{B^T B}{\|B\|} \right) + \eta I + \gamma I \right] < 1.$$

Then, the tolerable bound for uncertainties $f(x(k), k)$ *and* $f_1(x(k-d), k)$ *can be estimated as* $\eta + \gamma < 0.2395$*.*

4. **Conclusions.** In this paper, the robust stability for discrete uncertain systems with a time delay has been discussed. By choosing properly the positive matrix Q, a simple upper solution bound of the Lyapunov equation has been derived. Then, utilizing the Lyapunov equation approach associated with this upper solution bound, we have proposed several delay-independent conditions that assure the robust stability of the above systems. It is

seen that the present result of systems without uncertainties is better than an existing result. The feature of these conditions is that they do not involve any Lyapunov equation although the Lyapunov equation approach is utilized. Finally, illustrative examples and simulations have been given to demonstrate the merits and to confirm the correctness of the presented schemes.

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