Circuits Syst Signal Process (2011) 30:1417–1443
DOI 10.1007/s00034-011-9298-5

Robust Stability of Discrete Bilinear Uncertain Time-Delay Systems

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Received: 26 December 2009 / Revised: 18 January 2011 / Published online: 21 April 2011

Abstract This paper addresses the problem of robust stability for discrete homogeneous bilinear time-delay systems subjected to uncertainties. Two kinds of uncertainties are treated: (1) nonlinear uncertainties and (2) parametric uncertainties. For parametric uncertainties, we also discuss both unstructured uncertainties and interval matrices. By using the Lyapunov stability theorem associated with some linear algebraic techniques, several delay-independent criteria are developed to guarantee the robust stability of the overall system. One of the features of the newly developed criteria is its independence from the Lyapunov equation, although the Lyapunov approach is adopted. Furthermore, the transient response and the decay rate of the resulting systems are also estimated. In particular, the transient responses for the aforementioned systems with parametric uncertainties also do not involve any Lyapunov equation which remains unsolved. All the results obtained are also applied to solve the stability analysis of uncertain time-delay systems.

Keywords Homogeneous bilinear system · Transient response · Robust stability · Time delay · Uncertainty

1 Introduction

The bilinear system, a subsystem of nonlinear systems, exists naturally not only in many engineering areas such as nuclear, thermal, and chemical processes, but also in

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physical systems such as those in biology, socio-economics, ecology, and so on. During the past decades, the study of bilinear systems has become prominent. A number of works have been devoted to research on stability analysis and design of stabilization controllers for bilinear systems [1–6, 8–10, 14–16, 21, 24, 33]. In practice, due to the transmission of information, natural properties of system elements, computation of variables, etc., time delays exist in real-life systems [13, 18, 25] and are always a source of instability of feedback systems. Furthermore, when modeling an engineering system, system uncertainties occur due to the use of an approximate system model for the purpose of simplicity, data errors for evaluation, changes in environment conditions, aging, etc. Thus, both time delays and uncertainties should be estimated when modeling a system. Therefore, a great number of approaches for analyzing stability and/or designing controllers for time-delay systems with/without uncertainties have been developed in the literature [22, 29–32]. Furthermore, research regarding bilinear systems with time delay has also drawn great interest [11, 12, 20, 33]. Besides, several works have focused on the study of uncertain bilinear systems [7, 27]. However, upon surveying the literature, it seems that none of the existing results for discrete bilinear systems has considered both the system uncertainties and time delay(s). Therefore, this paper addresses the problem of robust stability for bilinear homogeneous time-delay systems subjected to uncertainties. Two kinds of uncertainties are treated: (1) nonlinear uncertainties and (2) parametric uncertainties. For the latter, both unstructured uncertainties and interval matrices are discussed. By using the Lyapunov equation approach, several delay-independent stability criteria are derived for the mentioned systems. However, an interesting consequence is that the criteria obtained do not involve any Lyapunov equation. The transient behavior and the decay rate of the overall system are also estimated. They are also independent of any Lyapunov equation. We also apply the presented schemes to solve the stability problem for (uncertain) bilinear systems and (uncertain) time-delay systems. Comparisons between the results obtained and those that have appeared in the literature are also made. Finally, numerical examples and computer simulations are given to demonstrate the applicability of the proposed schemes.

The organization of this paper is as follows. In Sect. 2, system models are presented and several delay-independent criteria that guarantee the robust stability of the mentioned systems are developed by using the Lyapunov equation approach associated with linear algebraic techniques. Transient responses and decay rates of the resulting systems are also derived. Then, the results obtained are applied to discuss a similar study for uncertain bilinear systems and uncertain time-delay systems. Furthermore, several numerical examples associated with computer simulations that demonstrate the application of the obtained results are given in Sect. 3. The conclusions are made in the last section.

The following symbols are used in this paper. Symbols $\mathbb{R}$, A^T, $\lambda_1(A)$, $\lambda_m(A)$, $x^T(k)$, $\|x(k)\|$, and $\|A\|$, respectively, refer to the real number field, the transpose of matrix A, the maximal eigenvalue of a symmetric matrix A, the minimal eigenvalue of a symmetric matrix A, transpose of vector $x(k)$, norm of vector $x(k)$ with $\|x(k)\| = (x^T(k)x(k))^{1/2}$, and induced norm of matrix A with $\|A\| = \lambda_1(A^T A)^{1/2}$. Symbols $\sigma_1(A)$ and $\sigma_n(A)$ are the maximal and minimal singular values of a matrix A, respectively. For a scalar a, $|a|$ denotes the modulus of scalar a. Furthermore, the modulus matrix of matrix A means $[|A|] = \{|a_{ij}|\}$ where $A = \{a_{ij}\}$, and $[|A|] \le B$ means $|a_{ij}| \le b_{ij}$ where $B = [b_{ij}]$.

2 Main Results

2.1 Nonlinear Uncertainties

Consider the following discrete homogeneous bilinear uncertain time-delay system:

$$x(k+1) = Ax(k) + A_1 x(k-d) + \sum_{i=1}^{m} u_i(k) B_i x(k) + B_{i1} x(k-d) + f\big(x(k), k\big) + f_1\big(x(k-d), k\big), \tag{1}$$

$$x(k) = \Phi(k), \quad k \in [-d, 0] \tag{2}$$

where $x(\cdot) \in \mathbb{R}^n$ represents the state vector, $u_i(\cdot) \in \mathbb{R}$, $i = 1, 2, \dots, m$, are inputs, integer $d > 0$ denotes the delay, $\Phi(k)$ is a known time function, A, A_1, B_i, and B_{i1} are constant matrices with appropriate dimensions, and $f(x(k), k)$ and $f_1(x(k-d), k)$ are nonlinear uncertainties with the following properties:

$$\big\| f\big(x(k), k\big)\big\| \le \varepsilon \big\|x(k)\big\| \quad \text{with } f(0, k) = 0, \tag{3}$$

$$\big\| f_1\big(x(k-d), k\big)\big\| \le \varepsilon_1 \big\|x(k-d)\big\| \quad \text{with } f_1(0, k) = 0, \tag{4}$$

where ε and ε_1 are positive constants.

Define

$$\max_k \big|u_i(k)\big| = U_i, \quad i = 1, 2, \dots, m. \tag{5}$$

Before deriving the robust stability criteria, we first review the following result.

Definition 1 The homogeneous bilinear uncertain time-delay system (1) is said to be exponentially stable with a decay rate ρ, $0 < \rho < 1$, if there exists a constant $c \ge 1$ such that

$$\big\|x(k)\big\| \le c \sup_{s \in [-d,0]} \big\|x(s)\big\| \rho^k, \quad k \ge 0. \tag{6}$$

Lemma 1 (Lee [17]) *For the discrete Lyapunov equation*

$$A^T P A - P = -Q, \tag{7}$$

where Q is a given positive definite matrix and A is a discrete stable matrix, the unique positive solution P satisfies

$$P \ge \frac{\lambda_n(Q)}{1 - \sigma_n^2(A)} A^T A + Q. \tag{8}$$

Furthermore, if $\sigma_1(A) < 1$, *then the solution* P *has the upper bound*

$$P \le \frac{\lambda_1(Q)}{1-\sigma_1^2(A)} A^T A + Q. \tag{9}$$

Note that $\sigma_1(A) = \|A\|$. Furthermore, we define the following symbols for convenience:

$$E \equiv \sum_{i=1}^{m} u_i(k) B_i, \tag{10}$$

$$E_1 \equiv \sum_{i=1}^{m} u_i(k) B_{i1}, \tag{11}$$

$$\beta \equiv \sum_{i=1}^{m} U_i \|B_i\|, \tag{12}$$

$$\beta_1 \equiv \sum_{i=1}^{m} U_i \|B_{i1}\|. \tag{13}$$

Utilizing the Lyapunov equation approach and Lemma 1, we derive the following criterion.

Theorem 1 *The uncertain bilinear time-delay system* (1) *is robustly stable if inputs* $u_i(k)$ *are designed to satisfy the condition*

$$\|A\|^2 + y + 2\sqrt{y\left[\|A\|^2 + \left(\|A\| + \beta + \varepsilon\right)^2\right]} < 1, \tag{14}$$

where the positive constant y *is defined as*

$$y \equiv (\beta+\varepsilon)^2 + \left(\|A_1\| + \beta_1 + \varepsilon_1\right)^2 \tag{15}$$

and positive constants ε *and* β, *respectively, are defined by* (3) *and* (12). *Furthermore, the transient behavior of the state of the overall system can be estimated as*

$$\|x(k)\| \le \sqrt{\frac{\|P\|(1+d\eta)}{\lambda_m(P)}} \sup_{s\in[-d,0]} \|x(s)\| \left[\sqrt{1 + \frac{q(\varphi-1)}{\lambda_m(P)[1-(1+\alpha)\|A\|^2]}}\right]^k, \tag{16}$$

where the positive definite matrix P *satisfies the Lyapunov equation*

$$(1+\alpha)A^T P A - P = -qI, \tag{17}$$

where q *is a given positive constant, the positive constants* φ *and* η, *respectively, are defined by*

$$\varphi \equiv (1+\alpha)\|A\|^2 + \left(1+\frac{1}{\alpha}\right)\left[(\beta+\varepsilon)^2 + \left(\|A_1\| + \beta_1 + \varepsilon_1\right)^2\right] + \alpha\left(\|A\| + \beta + \varepsilon\right)^2, \tag{18}$$

$$\eta \equiv \left(1+\frac{1}{\alpha}\right)\left(\|A_1\|+\beta_1+\varepsilon_1\right)^2 \tag{19}$$

and constant $\alpha > 0$ *is selected so that*

$$\frac{1-\|A\|^2-y-\sqrt{(\|A\|^2+y-1)^2-4y(\|A\|^2+z)}}{2(\|A\|^2+z)} < \alpha < \frac{1-\|A\|^2-y+\sqrt{(\|A\|^2+y-1)^2-4y(\|A\|^2+z)}}{2(\|A\|^2+z)} \tag{20}$$

with

$$z \equiv \left(\|A\|+\beta+\varepsilon\right)^2. \tag{21}$$

From (16), *the decay rate* ρ *is*

$$\rho \equiv \sqrt{1+\frac{q(\varphi-1)}{\lambda_m(P)[1-(1+\alpha)\|A\|^2]}}. \tag{22}$$

Proof For (20), inequality (14) implies that constant α is positive. Furthermore, it is easy to prove that the selection (20) for α results in $\varphi < 1$ where φ is defined by (18). This means that the following inequality must be met:

$$(1+\alpha)\|A\|^2 < 1. \tag{23}$$

Due to the fact $|\lambda(A)| \le \|A\|$ [28], (23) infers that $\sqrt{\alpha+1}A$ is stable and that the Lyapunov equation (17) is satisfied. In light of Lemma 1, we have

$$P \le \frac{q}{1-(1+\alpha)\|A\|^2}(1+\alpha)A^TA+qI. \tag{24}$$

Now, a Lyapunov function for the bilinear system (1) is constructed as

$$V\big(x(k)\big) = x^T(k)Px(k)+\left(1+\frac{1}{\alpha}\right)\|P\|\left(\|A_1\|+\beta_1+\varepsilon_1\right)^2\sum_{j=1}^{d}x^T(k-j)x(k-j), \tag{25}$$

where the positive definite matrix P satisfies (17). For convenience, we use V, x, x_d, f, and f_1 to replace $V(x(k)), x(k), x(k-d), f(x(k),k)$, and $f_1(x(k-d),k)$, respectively, in the following and later descriptions. Taking the forward difference for the Lyapunov function (25) results in

$$\begin{aligned}\Delta V &= x^T(k+1)Px(k+1)-x^TPx \\ &\quad + \left(1+\frac{1}{\alpha}\right)\|P\|\left(\|A_1\|+\beta_1+\varepsilon_1\right)^2\left(x^Tx-x_d^Tx_d\right) \\ &= (Ax+A_1x_d+Ex+E_1x_d+f+f_1)^TP(Ax+A_1x_d+Ex+E_1x_d \\ &\quad +f+f)-x^TPx\end{aligned}$$

$$
\begin{aligned}
&\quad + \left(1+\frac{1}{\alpha}\right)\|P\|\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2\big(x^T x - x_d^T x_d\big)\\
&= x^T\big(A^T P A - P\big)x + x^T A^T P(Ex+f)\\
&\quad + x^T(A+E+f)^T P\big[(A_1+E_1)x_d+f_1\big]\\
&\quad + \big(x^T E^T + f^T\big)PAx + \big(x^T E^T + f^T\big)P(Ex+f)\\
&\quad + \big[x_d^T(A_1+E_1)^T + f_1^T\big]P\big[(A+E)x+f\big]\\
&\quad + \big[x_d^T(A_1+E_1)^T + f_1^T\big]P\big[(A_1+E_1)x_d+f_1\big]\\
&\quad + \left(1+\frac{1}{\alpha}\right)\|P\|\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2\big(x^T x - x_d^T x_d\big). \qquad (26)
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\big(x^T E^T + f^T\big)P(Ex+f) &\le \|P\|\|Ex+f\|^2 \le \|P\|\big(\|E\|\|x\|+\|f\|\big)^2\\
&\le \|P\|\left(\sum_{i=1}^{m}\big|u_i(k)\big|\|B_i\|\|x\|+\varepsilon\|x\|\right)^2\\
&\le \|P\|(\beta+\varepsilon)^2\|x\|^2 = \|P\|(\beta+\varepsilon)^2 x^T x, \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
&\big[x_d^T(A_1+E_1)^T + f_1^T\big]P\big[(A_1+E_1)x_d+f_1\big]\\
&\quad \le \|P\|\big\|(A_1+E_1)x_d+f_1\big\|^2 \le \|P\|\left[\left(\|A_1\|+\sum_{i=1}^{m}\big|u_i(k)\big|\|B_{i1}\|\right)\|x_d\|+\|f_1\|\right]^2\\
&\quad \le \|P\|\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2 x_d^T x_d, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
&x^T A^T P(Ex+f) + \big(x^T E^T + f\big)PAx\\
&\quad \le \alpha x^T A^T PAx + \frac{1}{\alpha}\big(x^T E^T + f^T\big)P(Ex+f)\\
&\quad \le \alpha x^T A^T PAx + \frac{1}{\alpha}\|P\|(\beta+\varepsilon)^2 x^T x, \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
&\big[x_d^T(A_1+E_1)^T + f_1^T\big]P\big[(A+E)x+f\big]\\
&\qquad + \big[x^T(A+E)^T + f^T\big]P\big[(A_1+E_1)x_d+f_1\big]\\
&\quad \le \alpha x^T(A+E+f)^T P(A+E+f)x + \frac{1}{\alpha}x_d^T(A_1+E_1)^T P(A_1+E_1)x_d\\
&\quad \le \alpha\|P\|\big(\|A\|+\beta+\varepsilon\big)^2 x^T x + \frac{1}{\alpha}\|P\|\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2 x_d^T x_d. \qquad (30)
\end{aligned}
$$

Substituting the above inequalities into (26) leads to

$$
\begin{aligned}
\Delta V &\le x^T\big[(1+\alpha)A^T PA - P\big]x + \frac{1}{\alpha}\|P\|(\beta+\varepsilon)^2 x^T x + \alpha\|P\|\big(\|A\|+\beta+\varepsilon\big)^2 x^T x\\
&\quad + \frac{1}{\alpha}\|P\|\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2 x_d^T x_d
\end{aligned}
$$

$$
\begin{aligned}
&+\|P\|(\beta+\varepsilon)^2x^Tx+\|P\|\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2x_d^Tx_d \\
&+\left(1+\frac{1}{\alpha}\right)\|P\|\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2\big(x^Tx-x_d^Tx_d\big) \\
=&\left[-qI+\left(1+\frac{1}{\alpha}\right)\|P\|\left[(\beta+\varepsilon)^2+\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2\right]\right. \\
&\left.+\alpha\|P\|\big(\|A\|+\beta+\varepsilon\big)^2\right]x^Tx, \qquad (31)
\end{aligned}
$$

where the relations (5), (12), (13), and (17) are used. From Lemma 1 and (24), one obtains

$$\|P\|\le\left\|\frac{q}{1-(1+\alpha)\|A\|^2}(1+\alpha)A^TA+qI\right\|=\frac{1}{1-(1+\alpha)\|A\|^2}q. \qquad (32)$$

Substituting this inequality into (31) yields

$$
\begin{aligned}
\Delta V&\le\frac{q}{1-(1+\alpha)\|A\|^2}\left\{-1+(1+\alpha)\|A\|^2+\left(1+\frac{1}{\alpha}\right)y+\alpha z\right\}x^Tx \\
&=\frac{q(\varphi-1)}{1-(1+\alpha)\|A\|^2}\|x\|^2, \qquad (33)
\end{aligned}
$$

where constants y and z, respectively, are defined in (15) and (21). Thus, it is obvious that if condition (14) holds, then $\Delta V<0$ and the bilinear system (1) is robustly stable.

Furthermore, from the definition of the Lyapunov function (25), we have

$$
\begin{aligned}
&\lambda_m(P)\big\|x(k)\big\|^2 \\
&\le V\big(x(k)\big)\le\|P\|\left[\big\|x(k)\big\|^2+\left(1+\frac{1}{\alpha}\right)\|P\|\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2\sum_{j=1}^{d}\big\|x(k-j)\big\|^2\right]. \qquad (34)
\end{aligned}
$$

Then, from (34), we rewrite (33) as

$$\Delta V\le\frac{q(\varphi-1)}{1-(1+\alpha)\|A\|^2}\big\|x(k)\big\|^2\le\frac{q(\varphi-1)}{\lambda_m(P)[1-(1+\alpha)\|A\|^2]}V. \qquad (35)$$

Solving this inequality with respect to $V(x(k))$ results in

$$V\big(x(k)\big)\le V\big(x(0)\big)\left[1+\frac{q(\varphi-1)}{\lambda_m(P)[1-(1+\alpha)\|A\|^2]}\right]^k. \qquad (36)$$

From the definition (19) and the right-hand side of (34), we have

$$V\big(x(0)\big)\le\|P\|\left[\big\|x(0)\big\|^2+\eta\sum_{j=1}^{d}\big\|x(-j)\big\|^2\right]\le\|P\|(1+d\eta)\Big[\sup_{s\in[-d,0]}\big\|x(s)\big\|\Big]^2. \qquad (37)$$

Applying (37) and the left-hand side of (34) to (36), we obtain

$$\|x(k)\|^2 \le \frac{\|P\|(1+d\eta)}{\lambda_m(P)}\Big[\sup_{s\in[-d,0]}\|x(s)\|\Big]^2\left[1+\frac{q(\varphi-1)}{\lambda_m(P)[1-(1+\alpha)\|A\|^2]}\right]^k \tag{38}$$

which leads to

$$\|x(k)\| \le \sqrt{\frac{\|P\|(1+d\eta)}{\lambda_m(P)}}\sup_{s\in[-d,0]}\|x(s)\|\left[\sqrt{1+\frac{q(\varphi-1)}{\lambda_m(P)[1-(1+\alpha)\|A\|^2]}}\right]^k. \tag{39}$$

This is the transient response of the resulting system, and the decay rate ρ is hence obtained as (22). This completes the proof. □

Remark 1 It is noted that the decay rate ρ depends on the selections of positive constants q and α. Furthermore, an interesting consequence of this theorem is that the stability condition (14) is independent of the constants q and α although the Lyapunov equation (17) is used.

Remark 2 The application of Lemma 1 to the Lyapunov equation (17) gives

$$\lambda_m(P) \ge \lambda_m\left(\frac{q}{1-(1+\alpha)\sigma_n^2(A)}(1+\alpha)A^TA+qI\right) = \frac{q}{1-(1+\alpha)\sigma_n^2(A)}, \tag{40}$$

$$\begin{aligned}\lambda_m(P) &\le \lambda_m\left(\frac{q}{1-(1+\alpha)\sigma_1^2(A)}(1+\alpha)A^TA+qI\right) \le \frac{q(1+\alpha)\sigma_n^2(A)}{1-(1+\alpha)\|A\|^2}+q\\ &\le \frac{q}{1-(1+\alpha)\|A\|^2}.\end{aligned} \tag{41}$$

Furthermore, from the definition of φ, it is seen that

$$1-\varphi < 1-(1+\alpha)\|A\|^2. \tag{42}$$

Therefore,

$$\begin{aligned}1-\frac{q(1-\varphi)}{\lambda_m(P)[1-(1+\alpha)\|A\|^2]} &\ge 1-\frac{[1-(1+\alpha)\sigma_n^2(A)](1-\varphi)}{[1-(1+\alpha)\|A\|^2]}\\ &> (1+\alpha)\sigma_n^2(A) > 0\end{aligned} \tag{43}$$

and

$$1-\frac{q(1-\varphi)}{\lambda_m(P)[1-(1+\alpha)\|A\|^2]} \le 1-\frac{[1-(1+\alpha)\|A\|^2](1-\varphi)}{[1-(1+\alpha)\|A\|^2]} \le \varphi < 1. \tag{44}$$

Thus, it is obvious that the decay rate satisfies $0<\rho<1$.

2.2 Parametric Uncertainties

2.2.1 Unstructured Uncertainties

Now, consider the following discrete homogeneous bilinear time-delay system subjected to unstructured uncertainties:

$$x(k+1) = (A + \Delta A)x(k) + (A_1 + \Delta A_1)x(k-d) + \sum_{i=1}^{m} u_i(k)\big[(B_i + \Delta B_i)x(k) + (B_{i1} + \Delta B_{i1})x(k-d)\big], \tag{45}$$

where inputs $u_i(k)$ have the properties (5) and ΔA, ΔA_1, ΔB_i, and ΔB_{i1} denote the unstructured uncertainties with appropriate dimensions and have the following properties:

$$\|\Delta A\| \le \delta, \qquad \|\Delta A_1\| \le \xi, \qquad \|\Delta B_i\| \le \rho_i, \qquad \|\Delta B_{i1}\| \le \varsigma_i \tag{46}$$

where δ, ξ, ρ_i, and ς_i are positive constants.

Furthermore, we also define

$$\bar{A} \equiv A + \Delta A, \qquad \bar{A}_1 \equiv A_1 + \Delta A_1, \qquad \bar{E} \equiv \sum_{i=1}^{m} u_i(k)(B_i + \Delta B_i), \qquad \bar{E}_1 \equiv \sum_{i=1}^{m} u_i(k)(B_{i1} + \Delta B_{i1}), \tag{47}$$

$$\bar{\beta} \equiv \sum_{i=1}^{m} U_i\big(\|B_i\| + \rho_i\big), \tag{48}$$

$$\bar{\beta}_1 \equiv \sum_{i=1}^{m} U_i\big(\|B_{i1}\| + \varsigma_i\big), \tag{49}$$

$$\bar{x} \equiv \big(\|A\| + \delta\big)^2, \tag{50}$$

$$\bar{y} \equiv \beta^2 + \big(\|A_1\| + \bar{\beta}_1 + \xi\big)^2, \tag{51}$$

$$\bar{z} \equiv \big(\|A\| + \bar{\beta} + \delta\big)^2. \tag{52}$$

Then, for the robust stability of the system (45), we have the following result.

Theorem 2 *If inputs $u_i(k)$ of the system (45) are selected such that the following condition is met*:

$$\bar{x} + \bar{y} + 2\sqrt{\bar{y}(\bar{x} + \bar{z})} < 1, \tag{53}$$

where positive constants $\bar{x}$, $\bar{y}$, *and* $\bar{z}$ *are defined in* (50)–(52), *respectively, then the system* (45) *is robustly stable. The transient behavior of the resulting system is estimated as*

$$\begin{aligned} \|x(k)\| &\le \sqrt{\frac{1+d\bar{\eta}}{1-(1+\bar{\alpha})(\|A\|+\delta)^2}} \\ &\quad\times \sup_{s\in[-d,0]}\|x(s)\|\left[\sqrt{1+\frac{\bar{\varphi}-1}{[1-(1+\bar{\alpha})(\|A\|+\delta)^2]}}\right]^k, \end{aligned} \tag{54}$$

where the positive constants $\bar{\eta}$ *and* $\bar{\varphi}$, *respectively, are defined as*

$$\bar{\eta} \equiv \left(1+\frac{1}{\bar{\alpha}}\right)\big(\|A_1\|+\xi+,\bar{\beta}_1\big)^2, \tag{55}$$

$$\bar{\varphi} \equiv (1+\bar{\alpha})\big(\|A\|+\delta\big)^2+\left(1+\frac{1}{\bar{\alpha}}\right)\big[\bar{\beta}^2+\big(\|A_1\|+\xi+\bar{\beta}_1\big)^2\big]+\bar{\alpha}\big(\|A\|+\delta+\bar{\beta}\big)^2 \tag{56}$$

and positive constant $\bar{\alpha}$ *is chosen such that*

$$\begin{aligned} &\frac{1-\bar{x}-\bar{y}-\sqrt{(\bar{x}+\bar{y}-1)^2-4\bar{y}(\bar{x}+z)}}{2(\bar{x}+\bar{z})} \\ &\quad<\bar{\alpha}<\frac{1-\bar{x}-\bar{y}+\sqrt{(\bar{x}+\bar{y}-1)^2-4\bar{y}(\bar{x}+\bar{z})}}{2(\bar{x}+\bar{z})}. \end{aligned} \tag{57}$$

Proof Condition (53) assures that constant $\bar{\alpha}$ is positive. Then the selection of (57) implies $\bar{\varphi} < 1$. From the definition (56), $\bar{\varphi} < 1$ means $(1+\bar{\alpha})(\|A\|+\delta)^2 < 1$. Since

$$\|A+\Delta A\| \le \|A\|+\|\Delta A\| \le \|A\|+\delta, \tag{58}$$

the following Lyapunov equation is satisfied:

$$(1+\bar{\alpha})(A+\Delta A)^T P(A+\Delta A)-P=-qI, \tag{59}$$

where q is a given positive constant. We then choose the Lyapunov function for the system (45) as

$$V\big(x(k)\big)=x^T(k)Px(k)+\left(1+\frac{1}{\bar{\alpha}}\right)\|P\|\big(\|A_1\|+\xi+\bar{\beta}_1\big)^2\sum_{j=1}^{d}x^T(k-j)x(k-j), \tag{60}$$

where P is the positive solution of the Lyapunov equation (59).

Taking the forward difference for $V(x(k))$ results in

$$\begin{aligned} \Delta V &= x^T(k+1)Px(k+1)-x^T Px+\left(1+\frac{1}{\bar{\alpha}}\right)\|P\|\big(\|A_1\|+\xi+\bar{\beta}_1\big)^2 \\ &\quad\times\big(x^T x-x_d^T x_d\big) \end{aligned}$$

$$
\begin{aligned}
&= (\bar{A}x + \bar{A}_1 x_d + \bar{E}x + \bar{E}_1 x_d)^T P(\bar{A}x + \bar{A}_1 x_d + \bar{E}x + \bar{E}_1 x_d) - x^T P x \\
&\quad + \left(1 + \frac{1}{\bar{\alpha}}\right)\|P\|\big(\|A_1\| + \xi + \bar{\beta}_1\big)^2\big(x^T x - x_d^T x_d\big) \\
&= x^T\big[(A + \Delta A)^T P(A + \Delta A) - P\big]x + x^T \bar{A}^T P \bar{E} x \\
&\quad + x^T(\bar{A} + \bar{E})^T P(\bar{A}_1 + \bar{E}_1)x_d \\
&\quad + x^T \bar{E}^T P \bar{A} x + x^T \bar{E}^T P \bar{E} x + x_d^T(\bar{A}_1 + \bar{E}_1)^T P(\bar{A} + \bar{E})x \\
&\quad + x_d^T(\bar{A}_1 + \bar{E}_1)^T P(\bar{A}_1 + \bar{E}_1)x_d \\
&\quad + \left(1 + \frac{1}{\bar{\alpha}}\right)\|P\|\big(\|A_1\| + \xi + \bar{\beta}_1\big)^2\big(x^T x - x_d^T x_d\big).
\end{aligned}
\tag{61}
$$

From the assumption (4) and using the following facts:

$$
x^T \bar{A}^T P \bar{E} x + x^T \bar{E}^T P \bar{A} x \le \bar{\alpha} x^T (A + \Delta A)^T P(A + \Delta A)x + \frac{1}{\bar{\alpha}} x^T \bar{E}^T P \bar{E} x, \tag{62}
$$

$$
\begin{aligned}
&x^T(\bar{A} + \bar{E})^T P(\bar{A}_1 + \bar{E}_1)x_d + x_d^T(\bar{A}_1 + \bar{E}_1)^T P(\bar{A} + \bar{E})x \\
&\quad \le \bar{\alpha} x^T(\bar{A} + \bar{E})^T P(\bar{A} + \bar{E})x + \frac{1}{\bar{\alpha}} x_d^T(\bar{A}_1 + \bar{E}_1)^T P(\bar{A}_1 + \bar{E}_1)x_d,
\end{aligned}
\tag{63}
$$

(61) becomes

$$
\begin{aligned}
\Delta V \le x^T\bigg[&(1 + \bar{\alpha})(A + \Delta A)^T P(A + \Delta A) - P \\
&+ \left(1 + \frac{1}{\bar{\alpha}}\right)\bar{E}^T P \bar{E} + \bar{\alpha}(\bar{A} + \bar{E})^T P(\bar{A} + \bar{E})\bigg]x \\
&+ \left(1 + \frac{1}{\bar{\alpha}}\right)x_d^T(\bar{A}_1 + \bar{E}_1)^T P(\bar{A}_1 + \bar{E}_1)x_d \\
&+ \left(1 + \frac{1}{\bar{\alpha}}\right)\|P\|\big(\|A_1\| + \xi + \bar{\beta}_1\big)^2\big(x^T x - x_d^T x_d\big).
\end{aligned}
\tag{64}
$$

We also have

$$
\begin{aligned}
\bar{E}^T P \bar{E} &\le \|P\|\|\bar{E}\|^2 \le \|P\|\left\|\sum_{i=1}^m u_i(k)(B_i + \Delta B_i)\right\|^2 \\
&\le \|P\|\left[\sum_{i=1}^m \big|u_i(k)\big|\big(\|B_i\| + \|\Delta B_i\|\big)\right]^2 \le \|P\|\bar{\beta}^2,
\end{aligned}
\tag{65}
$$

$$
(\bar{A} + \bar{E})^T P(\bar{A} + \bar{E}) \le \|P\|\big(\|A\| + \|\Delta A\| + \|\bar{E}\|\big)^2 \le \|P\|\big(\|A\| + \delta + \bar{\beta}\big)^2, \tag{66}
$$

$$
\begin{aligned}
(\bar{A}_1 + \bar{E}_1)^T P(\bar{A}_1 + \bar{E}_1) &\le \|P\|\|\bar{A}_1 + \bar{E}_1\|^2 \le \|P\|\big(\|A_1\| + \|\Delta A_1\| + \|E_1\|\big)^2 \\
&\le \|P\|\big(\|A_1\| + \xi + \bar{\beta}_1\big)^2.
\end{aligned}
\tag{67}
$$

Then, substituting (65)–(67) into (64) results in

$$\Delta V \le \left[-qI + \left(1+\frac{1}{\bar{\alpha}}\right)\|P\|\left[\bar{\beta}^2 + \left(\|A_1\|+\xi+\bar{\beta}_1\right)^2\right] + \bar{\alpha}\|P\|\left(\|A\|+\delta+\bar{\beta}\right)^2\right]x^T x, \tag{68}$$

where relations (5) and (59) are utilized.

From the Lyapunov equation (59) and Lemma 1, one obtains

$$\begin{aligned}\|P\| &\le \left\|\frac{q}{1-(1+\bar{\alpha})\|A+\Delta A\|^2}(1+\bar{\alpha})(A+\Delta A)^T(A+\Delta A)+qI\right\| \\ &\le \frac{q}{1-(1+\bar{\alpha})(\|A\|+\delta)^2}(1+\bar{\alpha})\left(\|A\|+\delta\right)^2+q = \frac{1}{1-(1+\bar{\alpha})(\|A\|+\delta)^2}q.\end{aligned} \tag{69}$$

Applying (69) into (68) yields

$$\begin{aligned}\Delta V &\le \frac{q}{1-(1+\bar{\alpha})(\|A\|+\delta)^2}\Bigg[(1+\bar{\alpha})\left(\|A\|+\delta\right)^2 \\ &\quad + \left(1+\frac{1}{\bar{\alpha}}\right)\left[\beta^2+\left(\|A_1\|+\xi+\bar{\beta}_1\right)^2\right]\Bigg]x^T x + \left[\bar{\alpha}\left(\|A\|+\delta+\bar{\beta}\right)^2-1\right]x^T x \\ &= \frac{q}{1-(1+\bar{\alpha})(\|A\|+\delta)^2}(\bar{\varphi}-1)\left\|x(k)\right\|^2.\end{aligned} \tag{70}$$

Since $\bar{\varphi} < 1$, it shows that ΔV is negative, which guarantees the robust stability of (45).

From (60), we have

$$\begin{aligned}\lambda_m(P)\left\|x(k)\right\|^2 &\le V\left(x(k)\right) \\ &\le \|P\|\left[\left\|x(k)\right\|^2 + \left(1+\frac{1}{\bar{\alpha}}\right)\left(\|A_1\|+\xi+\bar{\beta}_1\right)^2\sum_{j=1}^{d}\left\|x(k-j)\right\|^2\right].\end{aligned} \tag{71}$$

Then, according to (71), we rewrite (70) as

$$\Delta V \le \frac{q(\bar{\varphi}-1)}{1-(1+\bar{\alpha})(\|A\|+\delta)^2}\left\|x(k)\right\|^2 \le \frac{q(\bar{\varphi}-1)}{\lambda_m(P)[1-(1+\bar{\alpha})(\|A\|+\delta)^2]}V. \tag{72}$$

Solving this inequality with respect to $V(x(k))$ results in

$$V\left(x(k)\right) \le V\left(x(0)\right)\left[1+\frac{q(\bar{\varphi}-1)}{\lambda_m(P)[1-(1+\bar{\alpha})(\|A\|+\delta)^2]}\right]^k. \tag{73}$$

From the right-hand side of (71), we have

$$V\left(x(0)\right) \le \|P\|\left[\left\|x(0)\right\|^2 + \left(1+\frac{1}{\bar{\alpha}}\right)\left(\|A_1\|+\xi+\bar{\beta}_1\right)^2\sum_{j=1}^{d}\left\|x(-j)\right\|^2\right]$$

$$\leq \|P\|(1+d)\left(1+\frac{1}{\bar{\alpha}}\right)\left(\|A_1\|+\xi+\bar{\beta}_1\right)^2\Big[\sup_{s\in[-d,0]}\big\|x(s)\big\|\Big]^2$$

$$= \|P\|(1+d\bar{\eta})\Big[\sup_{s\in[-d,0]}\big\|x(s)\big\|\Big]^2. \tag{74}$$

Then, applying (74) and the left-hand side of (71) to (73), we obtain

$$\big\|x(k)\big\|^2 \leq \frac{\|P\|[1+d\bar{\eta}]}{\lambda_m(P)}\Big[\sup_{s\in[-d,0]}\big\|x(s)\big\|\Big]^2\left[1+\frac{q(\bar{\varphi}-1)}{\lambda_m(P)[1-(1+\bar{\alpha})(\|A\|+\delta)^2]}\right]^k$$

which leads to

$$\big\|x(k)\big\| \leq \sqrt{\frac{\|P\|(1+d\bar{\eta})}{\lambda_m(P)}}\sup_{s\in[-d,0]}\big\|x(s)\big\|\left[\sqrt{1+\frac{q(\bar{\varphi}-1)}{\lambda_m(P)[1-(1+\bar{\alpha})(\|A\|+\delta)^2]}}\right]^k. \tag{75}$$

In light of (8) and the Lyapunov equation (59), the following inequality is satisfied:

$$P \geq \frac{q}{1-(1+\bar{\alpha})\sigma_n^2(A+\Delta A)}(1+\bar{\alpha})A^TA+qI \geq qI.$$

Then we have

$$\lambda_m(P) \geq q. \tag{76}$$

Therefore, the transient response (54) is obtained by substituting (76) and (69) into (75). Thus, the proof is completed. □

Remark 3 Note that the Lyapunov equation (59) is unsolvable. However, it is not necessary to solve this equation for the stability condition (53) and the transient response (54). Furthermore, condition (53) is also independent of positive constant $\bar{\alpha}$.

Remark 4 Proceeding in a way similar to that proposed in Remark 2, it can be shown that the decay rate $\bar{\rho}$ satisfies the following inequality:

$$0<\bar{\rho}\equiv\sqrt{1+\frac{q(\bar{\varphi}-1)}{\lambda_m(P)[1-(1+\bar{\alpha})(\|A\|+\delta)^2]}}<1.$$

2.2.2 *Interval Matrices*

Consider the following discrete interval bilinear time-delay system:

$$x(k+1)=A_Ix(k)+A_{I1}x(k-d)+\sum_{r=1}^{m}u_r(k)\big[B_{Ir}x(k)+B_{Ir1}x(k-d)\big], \tag{77}$$

where $x(\cdot)\in\mathbb{R}^n$, $u_r(\cdot)\in\mathbb{R}$, and $d>0$ are the same as those in system (1), $A_I=[a_{Iij}]$, $A_{I1}=[a_{I1ij}]$, $B_{Ir}=[b_{Irij}]$, and $B_{kId}=[b_{Ir1ij}]$ are interval matrices with appropriate dimensions and have the properties

$$\begin{aligned} &A_I \in N[U,V], \qquad A_{I1} \in N[U_1,V_1] \\ &\quad \text{with } U=[u_{ij}],\ V=[v_{ij}],\ U_1=[u_{1ij}],\ V_1=[v_{1ij}], \end{aligned} \tag{78}$$

$$\begin{aligned} &B_{Ir} \in N[E_r,F_r], \qquad B_{Ir1} \in N[E_{r1},F_{r1}] \\ &\quad \text{with } E_r=[e_{rij}],\ F_r=[f_{rij}],\ E_{r1}=[e_{r1ij}],\ F_{r1}=[f_{r1ij}] \end{aligned} \tag{79}$$

and $N[U,V]$, $N[U_1,V_1]$, $N[E_1,F_1]$, and $N[E_{r1},F_{r1}]$ are functions such that matrices A_I, A_{I1}, B_{Ir}, and B_{Ir1}, respectively, satisfy

$$\begin{aligned} &u_{ij} \le a_{Iij} \le v_{ij}, \qquad u_{1ij} \le a_{I1ij} \le v_{1ij}, \qquad e_{rij} \le b_{Irij} \le f_{rij}, \\ &e_{r1ij} \le b_{Ir1ij} \le f_{r1ij}, \quad i,j=1,2,\ldots,n. \end{aligned} \tag{80}$$

Define

$$\tilde{A} = [\tilde{a}_{ij}] \quad \text{with } \tilde{a}_{ij} \equiv \max\bigl(|u_{ij}|,|v_{ij}|\bigr), \tag{81}$$

$$\tilde{A}_1 = [\tilde{a}_{1ij}] \quad \text{with } \tilde{a}_{1ij} \equiv \max\bigl(|u_{1ij}|,|v_{1ij}|\bigr), \tag{82}$$

$$\tilde{B}_r = [\tilde{b}_{rij}] \quad \text{with } \tilde{b}_{rij} \equiv \max\bigl(|e_{rij}|,|f_{rij}|\bigr), \tag{83}$$

$$\tilde{B}_{r1} = [\tilde{b}_{r1ij}] \quad \text{with } \tilde{b}_{r1ij} \equiv \max\bigl(|e_{r1ij}|,|f_{r1ij}|\bigr), \tag{84}$$

$$\tilde{E} \equiv \sum_{r=1}^{m} u_r(k) B_{Ir}, \tag{85}$$

$$\tilde{E}_1 \equiv \sum_{r=1}^{m} u_r(k) B_{Ir1}, \tag{86}$$

$$\tilde{\beta} \equiv \sum_{r=1}^{m} U_r \|\tilde{B}_r\|, \tag{87}$$

$$\tilde{\beta}_1 \equiv \sum_{r=1}^{m} U_r \|\tilde{B}_{r1}\|. \tag{88}$$

From the definitions (81)–(84), one obtains

$$\bigl[|A_I|\bigr] \le \tilde{A}, \qquad \bigl[|A_{I1}|\bigr] \le \tilde{A}_1, \qquad \bigl[|B_{Ir}|\bigr] \le \tilde{B}_r, \quad \text{and} \quad \bigl[|B_{Ir1}|\bigr] \le \tilde{B}_{r1}. \tag{89}$$

Furthermore, due to the well-known fact that $\|A\| \le \|[|A|]\|$, we also have

$$\begin{aligned} &\|A_I\| \le \bigl\|\bigl[|A_I|\bigr]\bigr\| \le \|\tilde{A}\|, \qquad \|A_{I1}\| \le \|\tilde{A}_1\|, \qquad \|B_{Ir}\| \le \|\tilde{B}_r\|, \quad \text{and} \\ &\|B_{Ir1}\| \le \|\tilde{B}_{r1}\|. \end{aligned} \tag{90}$$

Theorem 3 *The interval bilinear time-delay system* (77) *is robustly stable if inputs* $u_r(k)$ *can be chosen such that the following condition is satisfied*:

$$\|\tilde{A}\|^2 + \tilde{\beta}^2 + \big(\|\tilde{A}_1\| + \tilde{\beta}_1\big)^2 + 2\sqrt{\big[\tilde{\beta}^2 + \big(\|\tilde{A}_1\| + \tilde{\beta}_1\big)^2\big]\big[\|\tilde{A}\|^2 + \big(\|\tilde{A}\| + \tilde{\beta}\big)^2\big]} < 1, \tag{91}$$

where positive constants $\tilde{\beta}$ *and* $\tilde{\beta}_1$ *are defined in* (87) *and* (88), *respectively.*

The transient behavior of the state can be estimated as

$$\|x(k)\| \le \sqrt{\frac{1 + d\tilde{\eta}}{1 - (1+\tilde{\alpha})\|\tilde{A}\|^2}} \sup_{s\in[-d,0]} \|x(s)\| \left[\sqrt{1 + \frac{\tilde{\varphi} - 1}{1 - (1+\tilde{\alpha})\|\tilde{A}\|^2}}\right]^k, \tag{92}$$

where the positive constants $\tilde{\eta}$ *and* $\tilde{\varphi}$, *respectively, are defined as*

$$\tilde{\eta} \equiv \left(1 + \frac{1}{\tilde{\alpha}}\right)\big(\|\tilde{A}_1\| + \tilde{\beta}_1\big)^2, \tag{93}$$

$$\bar{\varphi} \equiv (1+\tilde{\alpha})\|\tilde{A}\|^2 + \left(1 + \frac{1}{\tilde{\alpha}}\right)\big[\tilde{\beta}^2 + \big(\|\tilde{A}_1\| + \tilde{\beta}_1\big)^2\big] + \tilde{\alpha}\big(\|\tilde{A}\| + \tilde{\beta}\big)^2 \tag{94}$$

and positive constant $\tilde{\alpha}$ *is chosen so that*

$$\frac{1 - \tilde{x} - \tilde{y} - \sqrt{(\tilde{x}+\tilde{y}-1)^2 - 4\tilde{y}(\tilde{x}+\tilde{z})}}{2(\tilde{x}+\tilde{z})} < \tilde{\alpha} < \frac{1 - \tilde{x} - \tilde{y} + \sqrt{(\tilde{x}+\tilde{y}-1)^2 - 4\tilde{y}(\tilde{x}+\tilde{z})}}{2(\tilde{x}+\tilde{z})} \tag{95}$$

with $\tilde{x} \equiv \|\tilde{A}\|^2$, $\tilde{y} \equiv \tilde{\beta}^2 + (\|\tilde{A}_1\| + \tilde{\beta}_1)^2$, *and* $\tilde{z} \equiv (\|\tilde{A}\| + \tilde{\beta})^2$.

Proof If we choose constant $\tilde{\alpha}$ by (95), then condition (91) implies that $\tilde{\alpha}$ is positive and $\tilde{\varphi} < 1$, which infers $(1+\tilde{\alpha})\|\tilde{A}\|^2 < 1$. This means that $(1+\tilde{\alpha})\|A_I\|^2 < 1$ and the following Lyapunov equation is satisfied:

$$(1+\tilde{\alpha})A_I^T P A_I - P = -qI, \tag{96}$$

where q is a given positive constant. Furthermore, we choose the Lyapunov function for the system (77) as

$$V\big(x(k)\big) = x^T(k)Px(k) + \left(1 + \frac{1}{\tilde{\alpha}}\right)\|P\|\big(\|\tilde{A}_1\| + \tilde{\beta}_1\big)^2 \sum_{j=1}^{d} x^T(k-j)x(k-j), \tag{97}$$

where P is the positive solution of the Lyapunov equation (96).

Taking the forward difference of $V(x(k))$ and using (77) results in

$$\begin{aligned}\Delta V &= x^T(k+1)Px(k+1) - x^T Px + \left(1 + \frac{1}{\tilde{\alpha}}\right)\|P\|\big(\|\tilde{A}_1\| + \tilde{\beta}_1\big)^2\big(x^T x - x_d^T x_d\big)\\ &= (A_I x + A_{I1}x_d + \tilde{E}x + \tilde{E}_1 x_d)^T P(A_I x + A_{I1}x_d + \tilde{E}x + \tilde{E}_1 x_d) - x^T Px\end{aligned}$$

$$
\begin{aligned}
&+\left(1+\frac{1}{\tilde{\alpha}}\right)\|P\|\big(\|\tilde{A}_1\|+\tilde{\beta}_1\big)^2\big(x^Tx-x_d^Tx_d\big)\\
&=x^T\big(A_I^TPA_I-P\big)x+x^TA_I^TP\tilde{E}x+x^T(A_I+\tilde{E})^TP(A_{I1}+\tilde{E}_1)x_d\\
&\quad+x^T\tilde{E}^TPA_Ix+x^T\tilde{E}^TP\tilde{E}x+x_d^T(A_{I1}+\tilde{E}_1)^TP(A_I+\tilde{E})x\\
&\quad+x_d^T(A_{I1}+\tilde{E}_1)^TP(A_{I1}+\tilde{E}_1)x_d\\
&\quad+\left(1+\frac{1}{\tilde{\alpha}}\right)\|P\|\big(\|\tilde{A}_1\|+\tilde{\beta}_1\big)^2\big(x^Tx-x_d^Tx_d\big).
\end{aligned}
\tag{98}
$$

Since

$$
x^TA_I^TP\tilde{E}x+x^T\tilde{E}^TPA_Ix\le\tilde{\alpha}x^TA_I^TPA_Ix+\frac{1}{\tilde{\alpha}}x^T\tilde{E}^TP\tilde{E}x,\tag{99}
$$

$$
\begin{aligned}
&x^T(A_I+\tilde{E})^TP(A_{I1}+\tilde{E}_1)x_d+x_d^T(A_{I1}+\tilde{E}_1)^TP(A_I+\tilde{E})x\\
&\quad\le\tilde{\alpha}x^T(A_I+\tilde{E})^TP(A_I+\tilde{E})x+\frac{1}{\tilde{\alpha}}x_d^T(A_{I1}+\tilde{E}_1)^TP(A_{I1}+\tilde{E}_1)x_d
\end{aligned}
\tag{100}
$$

and

$$
\tilde{E}^TP\tilde{E}\le\|P\|\big\|[|\tilde{E}|]\big\|^2I\le\|P\|\left[\sum_{r=1}^m\big|u_r(k)\big|\big\|[|B_{Ir}|]\big\|\right]^2I\le\|P\|\tilde{\beta}^2I,\tag{101}
$$

$$
\begin{aligned}
(A_I+\tilde{E})^TP(A_I+\tilde{E})&\le\|P\|\big(\|A_I\|+\|\tilde{E}\|\big)^2I\le\|P\|\big(\big\||A_I|\big\|+\big\|[|\tilde{E}|]\big\|\big)^2I\\
&\le\|P\|\big(\|\tilde{A}\|+\tilde{\beta}\big)^2I,
\end{aligned}
\tag{102}
$$

$$
\begin{aligned}
(A_{I1}+\tilde{E}_1)^TP(A_{I1}+\tilde{E}_1)&\le\|P\|\|A_{I1}+\tilde{E}_1\|^2I\le\|P\|\big(\big\||A_{I1}|\big\|+\big\|[|\tilde{E}_1|]\big\|\big)^2I\\
&\le\|P\|\big(\|\tilde{A}_1\|+\tilde{\beta}_1\big)^2I,
\end{aligned}
\tag{103}
$$

(98) becomes

$$
\begin{aligned}
\Delta V&\le x^T\left[(1+\tilde{\alpha})A_I^TPA_I-P+\left(1+\frac{1}{\tilde{\alpha}}\right)\|P\|\tilde{\beta}^2+\alpha\|P\|\big(\|\tilde{A}\|+\tilde{\beta}\big)^2\right]x\\
&\quad+\left(1+\frac{1}{\tilde{\alpha}}\right)\|P\|\big(\|\tilde{A}_1\|+\tilde{\beta}_1\big)^2x_d^Tx_d\\
&\quad+\left(1+\frac{1}{\tilde{\alpha}}\right)\|P\|\big(\|\tilde{A}_1\|+\tilde{\beta}_1\big)^2\big(x^Tx-x_d^Tx_d\big)\\
&=\left[-qI+\left(1+\frac{1}{\tilde{\alpha}}\right)\|P\|\big[\tilde{\beta}^2+\big(\|\tilde{A}_1\|+\tilde{\beta}_1\big)^2\big]+\tilde{\alpha}\|P\|\big(\|\tilde{A}\|+\tilde{\beta}\big)^2\right]x^Tx,
\end{aligned}
\tag{104}
$$

where relation (96) is utilized.

From the Lyapunov equation (96) and Lemma 1, one obtains

$$\begin{aligned}
\|P\| &\le \left\| \frac{q}{1-(1+\tilde{\alpha})\|A_I\|^2}(1+\tilde{\alpha})A_I^T A_I + qI \right\| \\
&\le \frac{q}{1-(1+\tilde{\alpha})\|[|A_I|]\|^2}(1+\tilde{\alpha})\big\|[|A_I|]\big\|^2 + q \\
&= \frac{1}{1-(1+\tilde{\alpha})\|[|A_I|]\|^2} q \le \frac{1}{1-(1+\tilde{\alpha})\|\tilde{A}\|^2} q.
\end{aligned} \tag{105}$$

Applying this inequality into (104) yields

$$\begin{aligned}
\Delta V &\le \frac{q}{1-(1+\tilde{\alpha})\|\tilde{A}\|^2}\bigg[(1+\tilde{\alpha})\|\tilde{A}\|^2 + \left(1+\frac{1}{\tilde{\alpha}}\right)\big[\tilde{\beta}^2 + \big(\|\tilde{A}_1\| + \tilde{\beta}_1\big)^2\big] \\
&\quad + \tilde{\alpha}\big(\|\tilde{A}\| + \tilde{\beta}\big)^2 - 1\bigg] x^T x \\
&= \frac{q}{1-(1+\tilde{\alpha})\|\tilde{A}\|^2}(\tilde{\varphi}-1)\big\|x(k)\big\|^2.
\end{aligned} \tag{106}$$

Now, selecting the positive constant $\tilde{\alpha}$ by (95), condition (91) implies that $\tilde{\varphi} < 1$. Therefore, this shows that condition (91) assures the robust stability of (77).

Proceeding in a way similar to that proposed in the proof of Theorem 2, we can estimate the transient response and the decay rate of the resulting system. The details are omitted. Therefore, the proof is completed. □

Remark 5 For the stability test problem of the interval system (77), it is seen that the stability condition (91) does not involve the unsolvable Lyapunov equation (96) and the positive parameter $\tilde{\alpha}$. Besides, the transient response of the resulting system is also independent of $\tilde{\alpha}$.

Remark 6 The present schemes can also be applied to the robust stability analysis for homogeneous bilinear systems without a time delay. Let $d = 0$ in (1), (45), and (77). Then, they reduce to the following uncertain bilinear systems:

$$x(k+1) = Ax(k) + \sum_{i=1}^{m} u_i(k)B_i x(k) + f\big(x(k), k\big), \tag{107}$$

$$x(k+1) = (A+\Delta A)x(k) + \sum_{i=1}^{m} u_i(k)(B_i + \Delta B_i)x(k), \tag{108}$$

$$x(k+1) = A_I x(k) + \sum_{r=1}^{m} u_r(k)B_{Ir} x(k). \tag{109}$$

In light of the proofs of Theorems 1, 2, and 3, we obtain the following corollaries for the robust stability of the above systems.

Corollary 1 *The bilinear system* (107) *is robustly stable if inputs* $u_i(k)$ *are designed such that*

$$\|A\| + \beta + \varepsilon < 1 \tag{110}$$

where positive constants ε *and* β, *respectively, are defined by* (3) *and* (12). *Furthermore, the transient behavior of the overall system can be estimated as*

$$\|x(k)\| \le \sqrt{\frac{\|P\|}{\lambda_m(P)}}\|x(0)\|\left[\sqrt{1+\frac{q[(1+\alpha)\|A\|^2+(1+\frac{1}{\alpha})(\beta+\varepsilon)^2-1]}{\lambda_m(P)[1-(1+\alpha)\|A\|^2]}}\right]^k, \tag{111}$$

where the positive definite matrix P *satisfies the Lyapunov equation* (16) *and constant* $\alpha > 0$ *is selected by*

$$\frac{1-\|A\|^2-y-\sqrt{(\|A\|^2+y-1)^2-4y\|A\|^2}}{2\|A\|^2} < \alpha < \frac{1-\|A\|^2-y+\sqrt{(\|A\|^2+y-1)^2-4y\|A\|^2}}{2\|A\|^2} \tag{112}$$

with $y \equiv (\beta+\varepsilon)^2$.

Corollary 2 *If inputs* $u_i(k)$ *are chosen such that the following condition is satisfied*:

$$\|A\| + \delta + \bar{\beta} < 1, \tag{113}$$

where constants δ *and* $\bar{\beta}$ *are defined by* (46) *and* (48), *respectively, then the perturbed bilinear system* (108) *is robustly stable with the following transient response*:

$$\|x(k)\| \le \sqrt{\frac{1}{1-(1+\bar{\alpha})(\|A\|+\delta)^2}} \times \|x(0)\|\left[\sqrt{1+\frac{(1+\bar{\alpha})(\|A\|+\delta)^2+(1+\frac{1}{\bar{\alpha}})\bar{\beta}^2-1}{1-(1+\bar{\alpha})(\|A\|+\delta)^2}}\right]^k, \tag{114}$$

where positive constant $\bar{\alpha}$ *is chosen by*

$$\frac{1-\bar{x}-\bar{y}-\sqrt{(\bar{x}+\bar{y}-1)^2-4\bar{y}\bar{x}}}{2\bar{x}} < \bar{\alpha} < \frac{1-\bar{x}-\bar{y}+\sqrt{(\bar{x}+\bar{y}-1)^2-4\bar{y}\bar{x}}}{2\bar{x}} \tag{115}$$

with $\bar{x} \equiv (\|A\|+\delta)^2$ *and* $\bar{y} \equiv \beta^2$.

Corollary 3 *The interval bilinear system* (109) *is robustly stable if inputs* $u_i(k)$ *are selected such that the following condition is satisfied*:

$$\|\tilde{A}\| + \tilde{\beta} < 1. \tag{116}$$

Furthermore, the transient behavior of the state can be estimated as

$$\|x(k)\| \le \sqrt{\frac{1}{1-(1+\tilde{\alpha})\|\tilde{A}\|^2}}\|x(0)\|\left[\sqrt{1+\frac{(1+\tilde{\alpha})\|\tilde{A}\|^2+(1+\frac{1}{\tilde{\alpha}})\tilde{\beta}^2-1}{1-(1+\tilde{\alpha})\|\tilde{A}\|^2}}\right]^k, \tag{117}$$

where positive constant $\tilde{\alpha}$ *is chosen by*

$$\frac{1-\|\tilde{A}\|^2-\tilde{\beta}^2-\sqrt{(\|\tilde{A}\|^2+\tilde{\beta}^2-1)^2-4\tilde{\beta}^2\|\tilde{A}\|^2}}{2\|\tilde{A}\|^2} < \tilde{\alpha} < \frac{1-\|\tilde{A}\|^2-\tilde{\beta}^2+\sqrt{(\|\tilde{A}\|^2+\tilde{\beta}^2-1)^2-4\tilde{\beta}^2\|\tilde{A}\|^2}}{2\|\tilde{A}\|^2}. \tag{118}$$

Remark 7 By using the proposed schemes, it is seen that the stability conditions (110), (113), and (116) for the bilinear systems (107)–(109) are very concise.

Remark 8 Surveying the literature, it seems that the robust stability test problem of the discrete homogeneous perturbed bilinear systems (107)–(109) has not been solved yet. In order to compare with the results presented, we let $m=1$, $f(x(k),k)=0$, and $\Delta A=\Delta B_1=0$. Then (107) and (108) become the following model:

$$x(k+1)=Ax(k)+u_1(k)B_1x(k). \tag{119}$$

In the literature [10], two criteria for the stability of system (119) have been proposed as follows.

Theorem 4 (Gounaridis-minaidis and N. Kalouptsidis, [10]) *The system* (119) *is stable if and only if*

(i) *The eigenvalues of* A *have modules less than* 1 *and* B_1 *is nilpotent.*
(ii) $\operatorname{tr}P(k,i)=0, k=2,\dots,n, i=1,\dots,k-1$ *where* $P(k,i)$ *is defined by*

$$P(k,i)\equiv\sum_{\mu}A^{\mu_0}BA^{\mu_1}\dots BA^{\mu_i}. \tag{120}$$

$\mu_0,\dots,\mu_i$ *are non-negative integers and* $\mu=(\mu_0,\dots,\mu_i)$. *The above summation satisfies the relation* $\sum_{j=0}^{i}\mu_j=k-i$.

Theorem 5 (Gounaridis-minaidis and N. Kalouptsidis, [10]) *The system* (119) *is stable if and only if for any* $u_1(k)$ *and any given positive definite matrix* Q *there exists a unique positive definite matrix* $P(u(k))$ *such that*

$$\big(A+u_1(k)B_1\big)^TP\big(u(k)\big)\big(A+u_1(k)B_1\big)=-Q. \tag{121}$$

The matrix $P(u(k))$ *is defined as*

$$P\big(u(k)\big)=P_0+\sum_{i=1}^{q}u_{1i}P_i \tag{122}$$

whose matrix coefficients can be computed recursively by the formulas

$$A^T P_0 A - P_0 = -Q, \tag{123}$$

$$A^T P_{i+1} A - P_{i+1} = -R_i, \quad i = 0, 1, \ldots, q-1, \tag{124}$$

where

$$R_0 = A^T P_0 B + B^T P_0 A \tag{125}$$

and

$$R_j = A^T P_j B + B^T P_j A + B^T P_{j-1} B, \quad j = 1, 2, \ldots, q-1. \tag{126}$$

From Corollaries 1 and 2, the robust stability of system (119) is given as follows:

$$\|A\| + U_1 \|B_1\| < 1. \tag{127}$$

This is a sufficient condition. Those proposed in Theorems 4 and 5 are necessary and sufficient conditions. However, it is obvious that our result (127) is more concise. Furthermore, the transient behavior of the system (119) can also be estimated by Corollaries 1 and 2, but the schemes proposed in [10] cannot do this estimation.

Remark 9 Let $u_i(k) = u_r(k) = 0$. Systems (1), (45), and (77), respectively, become the following uncertain time-delay systems:

$$x(k+1) = Ax(k) + A_1 x(k-d) + f\big(x(k), k\big) + f_1\big(x(k-d), k\big), \tag{128}$$

$$x(k+1) = (A + \Delta A)x(k) + (A_1 + \Delta A_1)x(k-d), \tag{129}$$

$$x(k+1) = A_I x(k) + A_{I1} x(k-d). \tag{130}$$

Then, from the proofs of Theorems 1, 2, and 3, we can directly obtain the following corollaries for the robust stability of the above time-delay systems.

Corollary 4 *The uncertain time-delay system (128) is robustly stable if*

$$\|A\|^2 + \varepsilon^2 + \big(\|A_1\| + \varepsilon_1\big)^2 + 2\sqrt{\big[\varepsilon^2 + \big(\|A_1\| + \varepsilon_1\big)^2\big]\big[\|A\|^2 + \big(\|A\| + \varepsilon\big)^2\big]} < 1, \tag{131}$$

where positive constants ε and ε_1, respectively, are defined by (3) and (4). Furthermore, the transient behavior of this system can be estimated as

$$\|x(k)\| \le \sqrt{\frac{\|P\|[1 + d(1 + \frac{1}{\alpha})(\|A_1\| + \varepsilon_1)^2]}{\lambda_m(P)}} \times \sup_{s \in [-d,0]} \|x(s)\| \left[\sqrt{1 + \frac{q(\varphi - 1)}{\lambda_m(P)[1 - (1+\alpha)\|A\|^2]}}\right]^k, \tag{132}$$

where the positive definite matrix P *satisfies* (17), *the positive constant* φ *is defined by*

$$\varphi \equiv (1+\alpha)\|A\|^2 + \left(1+\frac{1}{\alpha}\right)y + \alpha z \tag{133}$$

and constant $\alpha > 0$ *is selected such that*

$$\frac{1-\|A\|^2 - y - \sqrt{(\|A\|^2+y-1)^2 - 4y(\|A\|^2+z)}}{2(\|A\|^2+z)} < \alpha < \frac{1-\|A\|^2 - y + \sqrt{(\|A\|^2+y-1)^2 - 4y(\|A\|^2+z)}}{2(\|A\|^2+z)} \tag{134}$$

with $y \equiv \varepsilon^2 + (\|A_1\|+\varepsilon_1)^2$ *and* $z \equiv (\|A\|+\varepsilon)^2$.

Corollary 5 *If*

$$\left(\|A\|+\delta\right)^2 + \left(\|A_1\|+\xi\right)^2 + 2\sqrt{\left(\|A_1\|+\xi\right)^2\left(\|A\|+\delta\right)^2} < 1 \tag{135}$$

then system (129) *is robustly stable. The transient behavior of this system can be estimated as*

$$\left\|x(k)\right\| \le \sqrt{\frac{1+d(1+\frac{1}{\bar{\alpha}})(\|A_1\|+\xi)^2}{1-(1+\bar{\alpha})(\|A\|+\delta)^2}} \times \sup_{s\in[-d,0]}\left\|x(s)\right\|\left[\sqrt{1+\frac{\bar{\varphi}-1}{[1-(1+\bar{\alpha})(\|A\|+\delta)^2]}}\right]^k, \tag{136}$$

where the positive constant $\bar{\varphi}$ *is defined as*

$$\bar{\varphi} \equiv (1+\bar{\alpha})\left(\|A\|+\delta\right)^2 + \left(1+\frac{1}{\bar{\alpha}}\right)\left[\left(\|A_1\|+\xi\right)^2\right] + \bar{\alpha}\left(\|A\|+\delta\right)^2 \tag{137}$$

and positive constant $\bar{\alpha}$ *is chosen by*

$$\frac{1-\bar{x}-\bar{y}-\sqrt{(\bar{x}+\bar{y}-1)^2 - 8\bar{y}\bar{x}}}{4\bar{x}} < \bar{\alpha} < \frac{1-\bar{x}-\bar{y}+\sqrt{(\bar{x}+\bar{y}-1)^2 - 8\bar{y}\bar{x})}}{4\bar{x}} \tag{138}$$

with $\bar{x} \equiv (\|A\|+\delta)^2$ *and* $\bar{y} \equiv (\|A_1\|+\xi)^2$.

Corollary 6 *The interval time-delay system* (130) *is robustly stable if the following condition is satisfied*:

$$\|\tilde{A}\|^2 + \|\tilde{A}_1\|^2 + 2\sqrt{2}\|\tilde{A}\|\|\tilde{A}_1\| < 1. \tag{139}$$

The transient behavior can be estimated as

$$\|x(k)\| \leq \sqrt{\frac{1+d(1+\frac{1}{\tilde{\alpha}})\|\tilde{A}_1\|^2}{1-(1+\tilde{\alpha})\|\tilde{A}\|^2}} \times \sup_{s\in[-d,0]}\|x(s)\|\left[\sqrt{1+\frac{(1+2\tilde{\alpha})\|\tilde{A}\|^2+(1+\frac{1}{\tilde{\alpha}})\|\tilde{A}_1\|^2-1}{1-(1+\tilde{\alpha})\|\tilde{A}\|^2}}\right]^k, \quad (140)$$

where positive constant $\tilde{\alpha}$ *is chosen by*

$$\frac{1-\|\tilde{A}\|^2-\|\tilde{A}_1\|^2-\sqrt{(\|\tilde{A}\|^2+\|\tilde{A}_1\|^2-1)^2-8\|\tilde{A}_1\|^2\|\tilde{A}\|^2}}{4\|\tilde{A}\|^2} < \tilde{\alpha} < \frac{1-\|\tilde{A}\|^2-\|\tilde{A}_1\|^2+\sqrt{(\|\tilde{A}\|^2+\|\tilde{A}_1\|^2-1)^2-8\|\tilde{A}_1\|^2\|\tilde{A}\|^2}}{4\|\tilde{A}\|^2}. \quad (141)$$

Remark 10 In [19], a robust stability condition for the time-delay system (128) has been established as follows.

Theorem 6 (Lee et al. [19]) *The discrete time-delay system* (128) *is robustly stable if there exists a positive constant* η *such that the following condition is satisfied*:

$$\|A\|^2+\frac{3}{\eta}\left(\|A_1\|^2+\varepsilon^2+\varepsilon_1^2\right)<\frac{1}{1+\eta}. \quad (142)$$

It is seen that the tightness between (131) and (142) cannot be made by any mathematical method. However, by utilizing the schemes proposed in [19], the transient behavior of the system cannot be estimated. Therefore, the result obtained here is more general than that given in [19].

Remark 11 If systems (128) and (129) are simplified to the following model:

$$x(k+1)=Ax(k)+A_1x(k-d), \quad (143)$$

then, for this case, stability conditions (131) and (135), respectively, become

$$\|A\|^2+\|A_1\|^2+2\sqrt{2}\|A\|\|A_1\|<1, \quad (144)$$

$$\|A\|+\|A_1\|<1. \quad (145)$$

It is seen that the condition (145) is sharper than (144). Furthermore, the condition (145) coincides with that presented in [23]. In [26], the following results have also been developed.

Theorem 7 (Stojanovic and Debeljkovic [26]) *If for any given positive matrix Q there exists a positive matrix P such that the following matrix equation is fulfilled*:

$$\left(1+\frac{\|A_1\|}{\|A\|}\right)A^T PA+\left(1+\frac{\|A\|}{\|A_1\|}\right)A_1^T PA_1-P=-Q, \tag{146}$$

then system (143) *with $\|A\|\neq 0$ and $\|A_1\|\neq 0$ is stable.*

Theorem 8 (Stojanovic and Debeljkovic [26]) *The discrete time-delay system* (143) *is stable if there exist positive matrices P and Q such that the following linear matrix inequality holds*:

$$\begin{bmatrix} Q-P & 0 & A^T P \\ 0 & -Q & A_1^T P \\ PA & PA_1 & -P \end{bmatrix} < 0. \tag{147}$$

Comparing to (146) and (147), it is obvious that (145) is more concise, but maybe somewhat conservative. Notice that by utilizing the methods proposed in [26], the transient behavior of the system cannot be estimated.

Remark 12 According to the results obtained, it is shown that all nominal systems must be stable. From the properties of the matrix norm, $\|A\| < 1$ implies that the matrix A is stable. But the reverse is not true. Therefore, this shows that the sufficient conditions obtained may be somewhat conservative. However, they are new and can easily be calculated.

3 Numerical Examples

The following examples are given to show the applicability of the results obtained in this paper.

Example 1 Consider the discrete uncertain bilinear time-delay system (1) with

$$m=1,\qquad A=\begin{bmatrix} -0.1 & -0.04 & 0 \\ 0 & -0.03 & -0.05 \\ 0 & 0.1 & 0.1 \end{bmatrix},\qquad A_1=\begin{bmatrix} 0.1 & 0 & 0 \\ -0.05 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$$B_1=\begin{bmatrix} 0.2 & -0.01 & 0.01 \\ 0 & 0 & 0 \\ 0 & 0.02 & -0.2 \end{bmatrix},\qquad B_{11}=\begin{bmatrix} 0.1 & 0 & -0.01 \\ 0 & 0.1 & 0 \\ -0.02 & 0 & 0.2 \end{bmatrix},\qquad d=2.$$

It is assumed that $\|f(x(k),k)\| \le 0.1\|x(k)\|$, $\|f_1(x(k-d),k)\| \le 0.1\|x(k-d)\|$, and input $u_1(t)$ is designed as $u_1(k)=0.5-0.7\sin 8k$. Then, it is seen that $U_1=1.2$. Since

$$\begin{aligned}&\|A\|^2+(\beta+\varepsilon)^2+\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2\\&\quad+2\sqrt{\big[(\beta+\varepsilon)^2+\big(\|A_1\|+\beta_1+\varepsilon_1\big)^2\big]\big[\|A\|^2+\big(\|A\|+\beta+\varepsilon\big)^2\big]}=0.9559,\end{aligned}$$

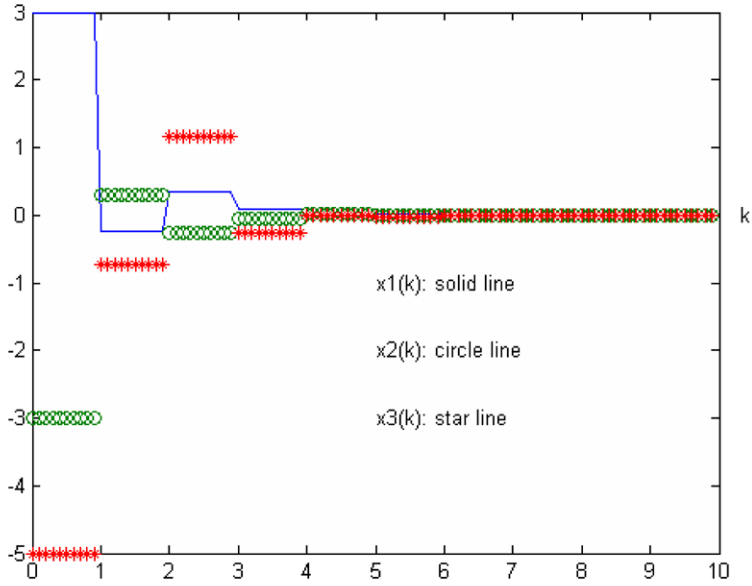

Fig. 1 The trajectories of states $x_i(k)$ for Example 1

according to Theorem 1, this system is robustly stable. Furthermore, in (16), if $q = 3$ and $\alpha = 1.1$, then the decay rate can be estimated as $\rho = 0.9765$. Let $f(x(k), k) = 0.1\cos 2kx(k)$, $f_1(x(k-d), k) = 0.1\cos kx(k-d)$, and $x(k) = [3\ -3\ -5]$ for $k \in [-2, 0]$. The simulation results for all states are shown in Fig. 1. Figure 2 shows the trajectory of the input $u_1(k)$. It is seen that all states are regulated to zero irrespective of the time delay and uncertainties.

Example 2 Consider the following bilinear uncertain time-delay system:

$$\begin{aligned} x(k+1) = {} & \left(\begin{bmatrix} -0.2 & -0.01 \\ 0 & 0.15 \end{bmatrix} + \Delta A\right) x(k) + \left(\begin{bmatrix} 0.1 & 0 \\ -0.1 & -0.02 \end{bmatrix} + \Delta A_1\right) x(t-d) \\ & + u_1(k)\left(\begin{bmatrix} -0.1 & 0.05 \\ 0 & -0.03 \end{bmatrix} + \Delta B_1\right) x(k) \\ & + u_1(k)\left(\begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} + \Delta B_{11}\right) x(k-d) \\ & + u_2(k)\left(\begin{bmatrix} -0.05 & 0.01 \\ 0 & -0.04 \end{bmatrix} + \Delta B_2\right) x(k) \\ & + u_2(k)\left(\begin{bmatrix} 0.06 & 0 \\ 0.03 & -0.03 \end{bmatrix} + \Delta B_{21}\right) x(k-d). \end{aligned} \tag{148}$$

For this case, we assume $\|\Delta A\| \le 0.1$, $\|\Delta A_1(t)\| \le 0.05$, $\|\Delta B_1\| \le 0.05$, $\|\Delta B_{11}(t)\| \le 0.08$, $\|\Delta B_2\| \le 0.02$, $\|\Delta B_{21}\| \le 0.05$, $U_1 = 1.1$, and $U_2 = 0.5$. In light of Theorem 2, the system (148) is robustly stable since

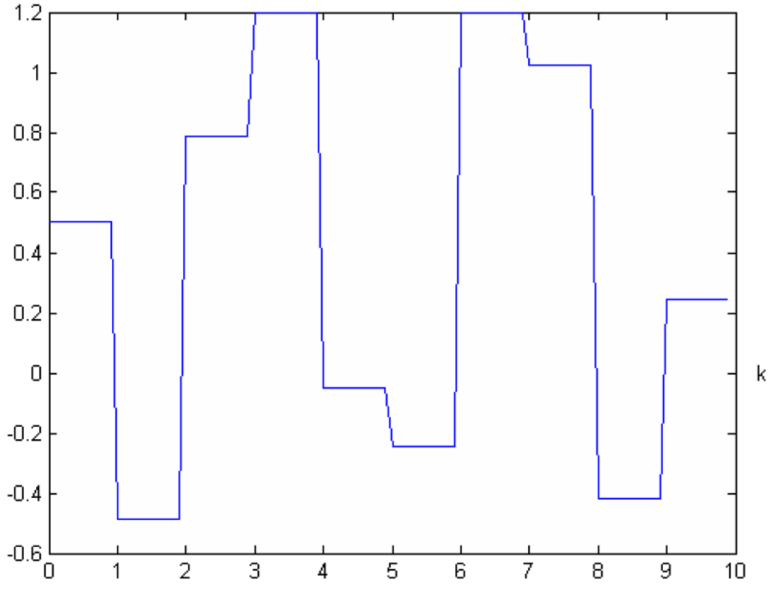

Fig. 2 The trajectory of the input $u_1(k)$ for Example 1

$$\left(\|A\|+\delta\right)^2+\bar{\beta}^2+\left(\|A_1\|+\xi+\bar{\beta}_1\right)^2$$
$$+2\sqrt{\left[\bar{\beta}^2+\left(\|A_1\|+\xi+\bar{\beta}_1\right)^2\right]\left[\left(\|A\|+\delta\right)^2+\left(\|A\|+\bar{\beta}+\delta\right)^2\right]}=0.9334.$$

Example 3 Consider the discrete bilinear interval system (77) with

$$m=1,\qquad A=\begin{bmatrix}-.05 & 0.01 & -0.05 & 0.03\\ 0 & & -.25 & -0.1\end{bmatrix},$$

$$A_1=\begin{bmatrix}-0.1 & 0.1 & -0.02 & 0.02\\ -0.01 & 0.03 & -0.05 & 0.1\end{bmatrix},\qquad B_1=\begin{bmatrix}-0.1 & 0.1 & & 0\\ 0.02 & 0.04 & -0.2 & -0.1\end{bmatrix},$$

$$B_{11}=\begin{bmatrix}0.05 & 0.15 & 0 & 0.01\\ -0.01 & 0.01 & -0.12 & -0.1\end{bmatrix},\qquad U_1=1.5.$$

According to the definitions (81)–(84), we have

$$\tilde{A}=\begin{bmatrix}0.05 & 0.05\\ 0 & 0.25\end{bmatrix},\qquad \tilde{A}_1=\begin{bmatrix}0.1 & 0.02\\ 0.03 & 0.1\end{bmatrix},\qquad \tilde{B}_1=\begin{bmatrix}0.1 & 0\\ 0.04 & 0.2\end{bmatrix},$$
$$\tilde{B}_{11}=\begin{bmatrix}0.15 & 0.01\\ 0.01 & 0.12\end{bmatrix}.$$

Then, the stability condition (91) is

$$\|\tilde{A}\|^2+\tilde{\beta}^2+\left(\|\tilde{A}_1\|+\tilde{\beta}_1\right)^2+2\sqrt{\left[\tilde{\beta}^2+\left(\|\tilde{A}_1\|+\tilde{\beta}_1\right)^2\right]\left[\|\tilde{A}\|^2+\left(\|\tilde{A}\|+\tilde{\beta}\right)^2\right]}$$
$$=0.8661<1$$

which guarantees the robustly stability of this bilinear interval system. Furthermore, the decay rate ρ can also be estimated as $\rho \geq 0.8487$ in which $\tilde{\alpha}$ is selected as $\tilde{\alpha} = 0.78$.

4 Conclusions

In this paper, robust stability for discrete homogeneous bilinear uncertain systems with a time delay has been discussed. By utilizing the Lyapunov equation approach, we have proposed several delay-independent conditions that assure the robust stability of the above systems. The feature of these conditions is that they do not involve any Lyapunov equation, although the Lyapunov equation approach is utilized. We also estimate the transient behavior and the decay rate of the resulting system. Furthermore, comparisons between the proposed results and those that have appeared in the literature have been made for the simplified system model. Finally, illustrative examples and simulations have been given to demonstrate the applicability and to confirm the correctness of the proposed schemes. In [24], it is shown that digital computer-controlled systems can be modeled by discrete bilinear systems. Considering time delay and uncertainty, they should become discrete bilinear uncertain time-delay systems. We believe that this work can be applied to stabilization controller design of the aforementioned systems. Therefore, finding an approach to select a desired controller to make the system robustly or exponentially stable would be part of a future work.

Acknowledgements The authors would like to thank the National Science Council, the Republic of China, for financial support of this research under the grant NSC97-2213-E-230-015. They would also like to express their sincere thanks to Prof. M.N.S. Swamy, Editor-in-Chief, for his suggestions, which have considerably improved the presentation of our paper.

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