A Modal Superposition Method for Vibration Responses of Atomic Force Microscope Cantilevers Using the Timoshenko Beam Model

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Abstract—Responses of the flexural vibration of an atomic force microscope (AFM) cantilever are evaluated using the Timoshenko beam theory. An analytical expression for the response of an AFM subjected to a sampling force with an excitation force of an arbitrarily chosen frequency is obtained with the help of the modal superposition method. In this investigation, the governing equations of the Timoshenko beam model with coupled differential equations expressed in terms of the flexural displacement and the bending angle were uncoupled to produce the fourth order equation. The effects of shear deformation were adopted to solve the dynamic model. A validity comparison for AFM modeling between the Timoshenko beam model and the Bernoulli-Euler beam model was conducted by using the ratios of the Young’s modulus to the shear modulus. Based on the results, the Bernoulli-Euler beam model for AFM applies to the small effects of transverse shear deformation, but not to ratios greater than 1000. Moreover, one can reduce the response at the end of AFM by decreasing the shear modulus when the frequencies of processing are far away from the modal frequencies, and by increasing the shear modulus when the frequencies of processing are close to the modal frequencies. Furthermore, an AFM with a large tip width and length is suitable for reducing the response at the end of the AFM.

Keywords—Modal superposition method; Timoshenko beam theory; Vibration response problem; Mode shape; Modal frequency; Atomic force microscope (AFM)

I. INTRODUCTION

In this investigation, the solution of the vibration response of an atomic force microscope cantilever is obtained by using the Timoshenko beam theory and the modal superposition method. In dynamic mode atomic force microscopy (AFM), information about the sample surface is obtained by monitoring the vibration parameters (e.g., amplitude or phase) of an oscillating cantilever which interacts with the sample surface. The atomic force microscope (AFM) was developed for producing high-resolution images of surface structures of both conductive and insulating samples in both air and liquid environments [1, 2, 3 and 4]. In addition, the AFM can be applied to nanolithography in micro/nano electromechanical systems (MEMS/NEMS) [5] and as a nanoindentation tester for evaluating mechanical properties [6]. Therefore, it is essential to precisely calculate the vibration response of AFM during the sampling process. In the last few years, there has been growing interest in the dynamic responses of the AFM cantilever. Horng [7] employed the modal superposition method to analyze the vibration responses of AFM cantilevers in tapping mode (TM) operated in a liquid and in air. Lin [8] derived the exact frequency shift of an AFM non-uniform probe with an elastically restrained root, subjected to van der Waals force, and proposed the analytical method to determine the frequency shift of an AFM V-shaped probe scanning the relative inclined surface in non-contact mode [9]. Girard et al. [10] studied dynamic atomic force microscopy operation based on high flexure modes vibration of the cantilever. Horng [11] developed an analytical solution to deal with the flexural vibration problem of AFM during a nanomachining process by using the modal superposition method.

The above studies considered the AFM cantilever as a Bernoulli-Euler beam model. The effects of transverse shear deformation and rotary inertia were assumed to be negligible in the analysis. However, for AFM-based direct mechanical nanomachining, the indentation and sampling of solid materials, such as polymer silicon and some metal surfaces, are performed. The effects of transverse shear deformation and rotary inertia in the vibration analysis should be taken into account for cantilevers whose cross-sectional dimensions are comparable to the lengths. Neglecting the effects of transverse shear deformation and rotary inertia in the vibration analysis may result in less accurate results. However, the solution of the vibration response obtained using the modal superposition method for AFM modeled as a Timoshenko beam, and the response of flexural vibration of a rectangular AFM cantilever which has large shear deformation effects, are absent from the literature.

In this paper, the response of flexural vibration of a rectangular AFM cantilever subjected to a sampling force is studied analytically by using the Timoshenko beam theory and the modal superposition method. Firstly, the governing equations of the Timoshenko beam model with coupled differential equations expressed in terms of the flexural displacement and the bending angle are uncoupled to produce the fourth order equation. Then, the sampling forces which are applied to the end region of the AFM by means of the tip, are transformed into an axial force, distributed transversal stress.
and bending stress. Finally, the response of the flexural vibration of a rectangular AFM cantilever subjected to a sampling force is solved using the modal superposition method. Moreover, a validity comparison for AFM modeling between the Timoshenko beam model and the Bernoulli-Euler beam model was conducted using the ratios of the Young’s modulus to the shear modulus.

II. ANALYSIS

The AFM cantilever moves down by a small amplitude (1-5 nm) when the cantilever tip processes a sample surface in contact mode. Therefore, the linear model can be used to describe the tip-sample interaction. The atomic force microscope cantilever, shown in Fig.1, is a small elastic beam with a length \( L \), thickness \( H \), width \( b \), and a tip with a width of \( w \) and length \( h \). \( x \) is the coordinate along the cantilever and \( \nu(x,t) \) is the vertical deflection in the \( x \)-direction, as shown in Fig.2. One end of the cantilever, at \( x = 0 \), is clamped, while the other end, from \( L - w \) to \( L \), has a tip.

When the sampling is in progress, the tip makes contact with the specimen, resulting in a vertical reaction force, \( F_y(t) \) and a horizontal reaction force, \( F_x(t) \), both of which functions of time \( t \). Assuming that the reaction forces act on the tip end, the product of the horizontal force and the tip length can form a bending stress on the bottom surface of the cantilever. The sampling system can be modeled as a flexural vibration motion of the cantilever. The motion is a function of mode shape and natural frequency, and its transverse displacement depends on time \( t \). When the beam support is constrained to be fixed and all other external influences are set to zero, we obtain the classical coupled Timoshenko-beam partial differential equations:

\[
\rho A \frac{\partial^2 \psi_x}{\partial t^2} - K_AG \frac{\partial^2 \psi_y}{\partial x^2} = 0
\]

\[
E I \frac{\partial^2 \psi_y}{\partial x^2} + K_AG \frac{\partial \psi_y}{\partial x} - \rho \frac{\partial^2 \psi_x}{\partial t^2} = 0
\]

where \( x \) is the distance along the center of the cantilever, \( \nu(x,t) \) is the transverse displacement, \( t \) is time, \( \psi(x,t) \) is the rotation of the neutral axis during bending, \( E \) is Young’s modulus, \( G \) is shear modulus, \( I \) is the area moment of inertia, \( \rho \) is the volume density, \( K \) is the shear factor (\( K = 5/6 \) for rectangular cross-section), and \( A \) is the rectangular cross-sectional area of the cantilever.

Equations (5) and (6) may be uncoupled to produce a fourth order equation in \( \nu(x,t) \). Considering the axial force effect, the classical uncoupled Timoshenko-beam partial differential equations can be written as:

\[
E I \frac{\partial^4 \nu(x,t)}{\partial x^4} + \rho A \frac{\partial^2 \nu(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \left[ \frac{N(t)}{E I} \frac{\partial \nu(x,t)}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \frac{K_G}{E I} \frac{\partial \nu(x,t)}{\partial y} \right] = p(x,t)
\]

where \( N(t) = F_y(t) \) is the axial force.

The mode-superposition analysis of a distributed-parameter system is equivalent to that of a discrete-coordinate system once the mode shapes and frequencies have been determined because in both cases, the amplitudes of the modal-response components are used as generalized coordinates in defining the response of the structure. In principle, an infinite number of these coordinates are available for a distributed-parameter system, since it has an infinite number of modes of vibration. Practically, however, only those modal components which provide significant contributions to the response need be considered [12 and 13]. The essential operation of the mode-superposition analysis is the transformation from the geometric displacement coordinates to the modal-amplitude or normal coordinates. For a one-dimensional system, this transformation is expressed as:

\[
\nu(x,t) = \sum_{n=1}^{\infty} \phi_n(x) Y_n(t) = \sum_{n=1}^{\infty} d_n(x,t)
\]

where \( d_n(x,t) \) is the response contribution of the \( n \)-th mode, \( Y_n(t) \) is the normal coordinate, and \( \phi_n(x) \) is the \( n \)-th mode shape of the AFM. In order to find the natural
frequencies and mode shapes, the following non-dimensional variables are defined:

\[ \xi = \frac{x}{L}, \quad b^2 = \frac{\rho M_i}{EI}, \quad r^2 = \frac{1}{AL}, \quad s^2 = \frac{EI}{KAGL^2} \]  

(9)

Here \( \xi \) is the non-dimensional length along the beam, and \( \omega \) is the radian frequency. Then, \( \phi_i(x) \) can be given by:

\[ \phi_i(\xi) = C \left[ \cosh b_i \alpha \xi - \frac{R_i - R_n}{R_n - R_R} \sinh b_i \alpha \xi - \cosh b_i \beta \xi + \frac{R_i - R_n}{R_n - R_R} \sin b_i \beta \xi \right] \]

(10)

where

\[ \{ \alpha, \beta \} = \left\{ \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}} \right\} \begin{pmatrix} (r^2 + s^2) / \alpha \beta \cos b_i \alpha \\ (r^2 + s^2) / \alpha \beta \sin b_i \alpha \end{pmatrix} \]

(11)

\[ R = \frac{\alpha^2 + s^2}{\alpha} \beta \sin b_i \alpha \]

(12)

\[ R_1 = (b_i / \alpha) \sinh b_i \alpha \]

(13)

\[ R_2 = (b_i / \alpha) \cosh b_i \alpha \]

(14)

\[ R_3 = (b_i / \alpha) \sin b_i \beta \]

(15)

\[ R_4 = (b_i / \alpha) \cos b_i \beta \]

(16)

and \( b_i \) are the non-dimensional natural frequencies, which can be obtained using the characteristic equation:

\[ \left( \frac{d^2}{dx^2} \right) R_{RR} - \left( \frac{d}{dx} \right) R_{RR} + R_{RR} - R_{RR} = 0 \]

(17)

where

\[ R'_1 = (\alpha^2 + s^2) / \alpha \beta \cos b_i \alpha \]

(18)

\[ R'_2 = (\alpha^2 + s^2) / \alpha \beta \sin b_i \alpha \]

(19)

\[ R'_3 = -(\beta^2 - s^2) / \beta \beta \cos b_i \beta \]

(20)

\[ R'_4 = -(\beta^2 - s^2) / \beta \beta \sin b_i \beta \]

(21)

Equation (8) simply states that any physically permissible displacement pattern can be modeled by superposing appropriate amplitudes of the vibration mode shapes for the structure. Substituting Eq. (8) into Eq. (7) and using orthogonally conditions gives

\[ S_i \frac{d^2 \xi_i(t)}{dt^2} + (M_{n} + T_{n} \frac{d \xi_i(t)}{dt} + \left[-G \xi_i(t) + \alpha \omega^2 M_{n} \xi_i(t)\right] \xi_i(t) = P_x(t) \]

(22)

where \( \omega_n \) is the \( n \)-th mode natural frequency of the AFM obtained using:

\[ \omega_n = \frac{b_i}{\rho M_i} \frac{EI}{\rho M_i} \]

(23)

\[ S_n, M_n, T_n \] and \( P_x \) are the generalized constants of the \( n \)-th mode, respectively, given by

\[ S_n = \left( \rho \frac{E}{KG} \right) \int_0^L \phi_i(x)^2 \ dx \]

(24)

\[ M_n = \left( \rho A \right) L_0 \int_0^L \phi_i(x)^2 \ dx \]

(25)

\[ T_n = \left( \rho L_0 \right) L_0 \int_0^L \frac{d \phi_i(x)}{dx} \ dx \]

(26)

\[ G_s(t) = \int_0^T \phi_i(x) \int_0^L \frac{d^2 \phi_i(x)}{dx^2} \ dx \]

(27)

\[ P_x(t) = \int_0^T \phi_i(x) \int_0^L \frac{d \phi_i(x)}{dx} \ dx \]

(28)

Using Eq. (3) and Eq. (4), Eq. (28) can be rewritten as

\[ \int_0^T \phi_i(x) \int_0^L \frac{d \phi_i(x)}{dx} \ dx = \frac{1}{c_s^2} F_x(t) \]

(29)

where

\[ c_s = \left[ \phi_i(x) \right]^{1/(L_x - L + w)} \]

(30)

Then, the Normal-Coordinate Response Equation, which is exactly the same equation considered for the discrete-parameter case, can be solved.

\[ \frac{d^2 \xi_i(t)}{dt^2} + (M_{n} + T_{n} \frac{d \xi_i(t)}{dt} + \left[-G \xi_i(t) + \alpha \omega^2 M_{n} \xi_i(t)\right] \xi_i(t) = c_s F_x(t) \]

(31)

Assuming a zero initial condition, with \( \xi(x,0) = 0 \), \( \dot{\xi}(x,0) = 0 \), \( \ddot{\xi}(x,0) = 0 \), and \( \dddot{\xi}(x,0) = 0 \), and provided that the sampling force \( F_x(t) \) is a series of harmonics, \( F_x(t) \) can be written as:

\[ F_x(t) = \sum_n F_n \sin(n \omega t) \]

(32)

When the \( j \)-th excitation frequency \( \omega_j \) is equal to the \( n \)-th natural frequency \( \omega_n \), the Runge-Kutta method is introduced to solve the above fourth-order system.

III. RESULTS AND DISCUSSION

The main goal of this study is to analyze the flexural vibration responses in nanoscale processing using atomic force microscopy modeled as a Timoshenko beam. To demonstrate the validity of the analytical solution, numerical computations were performed. The geometric and material parameters considered were as follows:

\[ E = 170 GPa, \quad \rho = 0.2898 \text{ km } / \text{ m }^3, \quad \rho = 2300 \text{ km } / \text{ m }^3, \quad L = 125 \mu \text{ m}, \quad b = 30 \mu \text{ m}, \quad H = 4.2 \mu \text{ m}, \quad h = 5 \mu \text{ m}, \quad 20 = 10^3 \text{ }, \quad m = 3 \text{,} \]

\[ F_x = 1000/(2i-1) \times 10^{-4} \]

The modulus-ratio \( REG \) defined as the ratio of \( E \) to \( G \) (i.e. \( REG = E / G \) ), is introduced to define the values of shear modulus \( G \) and to describe the effects of shear deformation. In this study, the flexural vibration responses at the end of the AFM were obtained using the contribution of the first five vibration modes. An non-dimensional response was used to normalize the static response as given in \( F_x(3EI) \), and \( \omega = (2i-1) \times \omega_n \) were set as the simulated values of the excitation frequency of the vertical sampling force. Thus, \( F_x(t) \) is taken as:

\[ F_x(t) = 100 \left( \sin(20t) + \frac{1}{3} \sin(3 \times 20t) + \frac{1}{5} \sin(5 \times 20t) \right) \left( 20^2 \right) \]

(33)

where \( r \) is the frequency ratio that can be used to describe the deviation between the excitation frequency and modal frequencies.

In order to investigate the effects of transverse shear deformation, the response histories at the end point of the AFM between different small and large modulus-ratios, with respect to excitation frequencies close to \( r = 0.9 \) the first
natural frequency, are shown in Fig.2. Figure 2 indicate that the responses are similar for the various modulus-ratios when they are more than 1000. Figure 2 reveals that the resonance effect occurs when the AFM has large modulus-ratios and the excitation frequencies are close to the modal frequencies.

IV. CONCLUSIONS

The modal superposition method and the Timoshenko beam theory were applied to an AFM-based nanoprocessing process to determine the flexural vibration responses at the end of the AFM. As expected, the Bernoulli-Euler beam model for AFM applies to the small effects of transverse shear deformation, but not for modulus-ratios greater than 1000. When modulus-ratios are greater than 1000, the Timoshenko beam model is the proper choice for simulating the flexural vibration responses of AFM.

References