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以 DSP 為核心的平面式倒單擺之 H_∞ 狀態回授干擾消除控制

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摘　要

本論文主要研究平面式倒單擺系統的控制問題。平面式倒單擺系統是一個非線性且不穩定的系統，因此很適合用來研究非線性與強健性控制的性能與適用性。因為平面式倒單擺的操作特性相當吸引學生的目光，所以普遍應用在教學實驗方面。一般機電整合的實驗與控制系統的實驗都有使用到平面式倒單擺系統。

本論文研究 H_∞ 強健狀態回授干擾抑制理論並將其應用在平面式倒單擺，使用此控制方法可以使平面式倒單擺穩定平衡。當手指觸碰干擾倒單擺時，倒單擺會很快消除干擾的作用，回到穩定平衡的狀態。平面式倒單擺有兩個平衡位置，一個是在中間位置，另一個是在頂點位置。在這兩個位置點，各有不同的線性狀態方程式組，H_∞ 控制方法也分別加以設計，使其達到強健控制的目的。

控制系統對於干擾與不確定性的強健性分析與控制一直以來都是主要的議題。對於大部分控制系統而言，若沒有干擾與不確定性，則不太需要用到回授控制。在控制工程界，多變數強健控制方法在1980年到2000年間一直都是重要的焦點，H_∞ 強健控制理論則是眾多學者努力的成果。本論文引用這方面的成果應用在平面式倒單擺系統，可以作為大學部或研究所學習控制系統或機電整合課程的學習教材或範例。

關鍵字：H_∞ 狀態回授、去除干擾、平面式倒單擺、數位信號處理器。

1. Introduction

The pendubot (pendulum robot) is a two-link planar robot, which first link (arm) is actuated and second link (pendulum) rotates freely. The pendubot is a benchmark system for under-actuated robot manipulators, i.e., they have fewer actuators than degrees of freedom. The pendubot is also used for research in nonlinear control. Many studies have been proposed to swing up and balance the pendubot. Spong and Block [1] applied the partial feedback linearization approach to swing up the pendulum and used a linear quadratic regulator (LQR) to balance the pendulum. Fantoni, Lozano and Spong [2] utilized energy based control to swing up and balance the pendulum by simulations. Zhang and Tarn [3] proposed hybrid control to balance the pendulum. The hybrid control is composed of discrete-time control and continue-time control simultaneously. Sanchez *et al.* [4] realized balance control of the pendubot via fuzzy control. Combining linear regulator theory with Takagi-Sugeno fuzzy technology, the tracking control is implemented. Craig *et al*. [5] implemented the arm-driven inverted pendulum system as a mechatronic system design case study. The arm-driven inverted pendulum is the same as the pendubot. Development of the swing-up and balancing controller and development of the embedded controller for the arm-driven inverted pendulum system are still continuations of their work. From another viewpoint, Tanaka *et al.* [6] realized acrobatic game of the pendubot by simulations. The acrobatic game is keeping the arm swinging periodically, while the pendulum maintains standing vertically.

The apparatus EMECS (Electro-Mechanical Engineering Control System) [7-9] used in this paper is supplied by TeraSoft Inc. The main feature of EMECS is an embedded control system. Because the digital controller of EMECS is the TMS320F2812 chip [10], EMECS can operate stand-alone. The dc motor is driven by the PWM port of TMS320F2812 chip [8-10]. The encoders of the dc motor and pendulum are linked to the QEP port of TMS320F2812 chip [8-10]. The H_∞ control theory has been an important topic in control engineering since 1980. By the endeavors of researchers all over the world, the H_∞ control theory has got a great achievement. In this paper, the H_∞ control is applied to this DSP-based pendubot. The main contribution of this paper is for the undergraduate control education. The pendubot is a kind of the inverted pendulums and an original nonlinear unstable system. The appropriate controls are necessary to maintain the robust stability. Hence, the pendubot is an excellent educational tool and impresses students. From hands-on experiments, students can study various control algorithms and realize how they work.

The H_∞ control is reviewed in Section 2. The system description is delineated in Section 3. The linear models of top and middle positions are derived in Section 4. The experimental results are shown in Section 5 and conclusions are described in Section 6.

2. Review of H_∞ Control

Consider the linear time-invariant system

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where x is the state vector and y is the output vector. A, B and C are system matrices. The transfer matrix from u to y is $H(s) = C(sI - A)^{-1}B$.

Lemma :

If there exists a real symmetric matrix X >0 and real numbers γ and δ such that

$$A^T X + XA + \gamma^{-2} XBB^T X + \delta^{-2} C^T C < 0 \tag{3}$$

then $\|H(j\omega)\|_\infty < \gamma\delta$. (4)

Proof:

Suppose inequality (3) is hold. Adding $j\omega X$ to and subtracting it from (3) can obtain

$$-\left(-j\omega I - A^T\right)X - X\left(j\omega I - A\right) + \gamma^{-2} XBB^T X + \delta^{-2} C^T C < 0 \tag{5}$$

Left multiplying $\gamma^{-1} B^T \left(-j\omega I - A^T\right)^{-1}$ and right multiplying $\gamma^{-1}\left(j\omega I - A\right)^{-1} B$ can reduce

$$-\Omega(j\omega) - \Omega^T(-j\omega) + \Omega^T(-j\omega)\Omega(j\omega) + \frac{1}{\gamma^2\delta^2} H^T(-j\omega)H(j\omega) < 0 \tag{6}$$

where $\Omega(j\omega) = \gamma^{-2} B^T X\left(j\omega I - A\right)^{-1} B$. From (6),

$$[I - \Omega(j\omega)]^* [I - \Omega(j\omega)] < I - \frac{1}{\gamma^2\delta^2} H^*(j\omega)H(j\omega) \tag{7}$$

We can imply

$$I - \frac{1}{\gamma^2\delta^2} H^*(j\omega)H(j\omega) > 0, \quad \forall\omega \tag{8}$$

and $\|H(j\omega)\|_\infty < \gamma\delta, \quad \forall\omega$ (9)

■

Now, the torque disturbance is added to system, (1) can be rewritten as

$$\dot{x} = Ax + Bu + Gw \tag{10}$$

The regulated output is

$$z = \begin{bmatrix} Cx \\ R^{1/2} u \end{bmatrix} \tag{11}$$

where $R > 0$, (A, B) is stabilizable and (A, C) is detectable. The symbol w is the disturbance. The state feedback controller

$$u = Kx \tag{12}$$

such that

$$\|T_{zw}(j\omega)\|_\infty = \left\| E\left(j\omega I - \tilde{A}\right)^{-1} G \right\|_\infty < \gamma \tag{13}$$

where $E = \begin{bmatrix} C \\ R^{1/2} K \end{bmatrix}$, and $\tilde{A} = A + BK$, $\gamma > 0$. From Lemma, there exists a positive real symmetry matrix X such that

$$(A + BK)^T X + X(A + BK) + \frac{1}{\gamma^2} XGG^T X + \begin{bmatrix} C^T & K^T R^{\frac{1}{2}T} \end{bmatrix} \begin{bmatrix} C \\ R^{1/2} K \end{bmatrix} < 0 \tag{14}$$

Substituting $XBR^{-1}B^T X$ can reduce

$$A^T X + XA + \gamma^{-2} XGG^T X + C^T C - XBR^{-1}B^T X + \left(K^T R^{\frac{1}{2}T} + XBR^{-\frac{1}{2}T}\right)\left(R^{\frac{1}{2}}K + R^{-\frac{1}{2}}B^T X\right) < 0 \quad (15)$$

If we choose the constant feedback gain as

$$K = -R^{-1}B^T X \quad (16)$$

Inequality (14) can be rewritten as

$$A^T X + XA + \gamma^{-2} XGG^T X + C^T C - XBR^{-1}B^T X < 0 \quad (17)$$

Hence if there exists a positive real symmetry matrix X satisfies the inequality (17), then $\tilde{A} = A - BR^{-1}B^T X$ is stable and $\left\| T_{zw}(j\omega) \right\|_\infty < \gamma$. The design procedure is given as follows.

Algorithm:

***Step 1*:** Choosing an arbitrary positive real number $\tilde{\gamma}$ smaller than the performance index, solve the algebraic Riccati equation

$$A^T X + XA + \tilde{\gamma}^{-2} XGG^T X + C^T C - XBR^{-1}B^T X + \varepsilon I = 0 \quad (18)$$

where ε =1.

***Step 2*:** If the solution X >0 of (18) exists, stop. If the solution does not exist, reduce the ε value and solve (18) again. If there is no solution for ε <<1, increase $\tilde{\gamma}$ value and go back to step 1.

3. System Description

Consider the DSP-based pendubot [7-9] as shown in Figures 1 and 2, the system variables and parameters are defined as follows:

m_a : mass of arm, m_p : mass of pendulum,

l_a : length of arm, c_a : center of gravity of arm, c_p : center of gravity of pendulum,

J_a : inertia of arm, J_p : inertia of pendulum,

θ_a : angular displacement of arm, θ_p : angular displacement of pendulum,

g : gravitational acceleration, τ_m : torque generated by the dc motor.

The dynamic equation of motion can be derived by the Euler-Lagrange equation [1-6] and is written as follows,

$$M(q)\ddot{q} + B(q,\dot{q})\dot{q} + G(q) = \begin{bmatrix} \tau_m \\ 0 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (19)$$

where $q=\begin{bmatrix}\theta_a\\ \theta_p\end{bmatrix}$ and $\begin{bmatrix}w_1\\ w_2\end{bmatrix}$ is the disturbance torques,

$$M(q)=\begin{bmatrix}P_1+P_2+2P_3\cos\theta_p & P_2+P_3\cos\theta_p\\ P_2+P_3\cos\theta_p & P_2\end{bmatrix},\quad B(q,\dot{q})=\begin{bmatrix}-P_3\dot{\theta}_p\sin\theta_p & -P_3(\dot{\theta}_a+\dot{\theta}_p)\sin\theta_p\\ P_3\dot{\theta}_a\sin\theta_p & 0\end{bmatrix},$$

$$G(q)=\begin{bmatrix}P_4\cos\theta_a+P_5\cos(\theta_a+\theta_p)\\ P_5\cos(\theta_a+\theta_p)\end{bmatrix},$$

and $P_1=J_a+m_ac_a^2+m_pl_a^2$, $P_2=J_p+m_pc_p^2$, $P_3=m_pl_ac_p$, $P_4=(m_ac_a+m_pl_a)g$,

$P_5=m_pc_pg$. The torque generated by the dc motor is

$$\tau_m=P_7v_a-P_6\dot{\theta}_a \qquad (20)$$

where $P_6=\dfrac{k_ik_b}{R_a}$, $P_7=\dfrac{k_i}{R_a}$, R_a is the armature resistance, v_a is the applied voltage, k_i is the torque constant and k_b is the back-emf constant.

For controller synthesis, we require a state-variable description of the pendubot. This is easily done by defining state variables $x_1=\theta_a$, $x_2=\theta_p$, $x_3=\dot{\theta}_a$ and $x_4=\dot{\theta}_p$, and the control input is $u=v_a$ to get

$$\dot{x}_1=x_3 \qquad (21)$$

$$\dot{x}_2=x_4 \qquad (22)$$

$$\dot{x}_3=\frac{1}{\Delta}\{P_2\Delta_1-(P_2+P_3\cos x_2)\Delta_2\} \qquad (23)$$

$$\dot{x}_4=\frac{1}{\Delta}\{(P_1+P_2+2P_3\cos x_2)\Delta_2-(P_2+P_3\cos x_2)\Delta_1\} \qquad (24)$$

where $\Delta=P_2(P_1+P_2+2P_3\cos x_2)-(P_2+P_3\cos x_2)^2$,

$\Delta_1=P_3x_3x_4\sin x_2+P_3x_4(x_3+x_4)\sin x_2-P_4\cos x_1-P_5\cos(x_1+x_2)-P_6x_3+P_7u+w_1$,

$\Delta_2=-P_3x_3^2\sin x_2-P_5\cos(x_1+x_2)+w_2$.

4. Linear Models

There are two control objectives investigated for the DSP-based pendubot. The first control objective is to stabilize the pendulum in the middle position equilibrium point, i.e., the arm is downward and the pendulum is upward. The second control objective is to stabilize the pendulum in the top position equilibrium point, i.e., the arm and the pendulum are all upward.

In order to design the linear controllers, the nonlinear model (21)-(24) needed to be linearized.

In the middle equilibrium position, the state condition is

$[x_1 \quad x_2 \quad x_3 \quad x_4]^T = \begin{bmatrix} \frac{-\pi}{2} & \pi & 0 & 0 \end{bmatrix}^T$ and the linear state equations are

$$\dot{x}_1 = x_3 \tag{25}$$

$$\dot{x}_2 = x_4 \tag{26}$$

$$\dot{x}_3 = \frac{1}{\Delta_L}\{(P_3P_5 - P_2P_4)x_1 + P_3P_5x_2 - P_2P_6x_3 + P_2P_7u + P_2w_1 + (P_3 - P_2)w_2\} \tag{27}$$

$$\dot{x}_4 = \frac{1}{\Delta_L}\{(P_1P_5 - P_3P_5 + P_2P_4 - P_3P_4)x_1 + P_5(P_1 - P_3)x_2 + (P_2 - P_3)P_6x_3 - (P_2 - P_3)P_7u + (P_3 - P_2)w_1 + (P_1 + P_2 - 2P_3)w_2\} \tag{28}$$

where $\Delta_L = P_2(P_1 + P_2 - 2P_3) - (P_2 - P_3)^2$. With the numerical values of physical parameters given in [7-9], the state equation of the linear system (25)-(28) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -20.4948 & 10.8502 & -0.0683 & 0 \\ 73.942 & 101.3689 & -0.0598 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3.9047 \\ 3.4166 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 273.4 & 239.2 \\ 239.2 & 4549.6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{29}$$

In the top equilibrium position, the state condition is $[x_1 \quad x_2 \quad x_3 \quad x_4]^T = \begin{bmatrix} \frac{\pi}{2} & 0 & 0 & 0 \end{bmatrix}^T$

and the linear state equations are

$$\dot{x}_1 = x_3 \tag{30}$$

$$\dot{x}_2 = x_4 \tag{31}$$

$$\dot{x}_3 = \frac{1}{\Delta_L}\{(P_2P_4 - P_3P_5)x_1 - P_3P_5x_2 - P_2P_6x_3 + P_2P_7u + P_2w_1 - (P_2 + P_3)w_2\} \tag{32}$$

$$\dot{x}_4 = \frac{1}{\Delta_L}\{(P_1P_5 + P_3P_5 - P_2P_4 - P_3P_4)x_1 + P_5(P_1 + P_3)x_2 + (P_2 + P_3)P_6x_3 - (P_2 + P_3)P_7u - (P_2 + P_3)w_1 + (P_1 + P_2 + 2P_3)w_2\} \tag{33}$$

With the numerical values of physical parameters given in [7-9], the state equation of the linear system (30)-(33) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 20.4948 & -10.8502 & -0.0683 & 0 \\ 32.9524 & 123.0693 & 0.1965 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3.9047 \\ -11.226 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 273.4 & -785.9 \\ 785.9 & 6599.9 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{34}$$

5. Experimental Results

The EMECS is utilized as the system hardware which is shown in figures 3 and 4. The CCS IDE (code composer studio integrated development environment) is exploited for controller software. The CCS includes a text-editor, a C-complier, an assembler, a debugger, and a download tool. Users can realize control theories by use of C-algorithms. Considering equation (19), $w=\begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ is uncertain direct disturbance torque which is imposed on the pendubot. The control objective is to reduce the bound $\left\|T_{zw}(j\omega)\right\|_\infty$, where the regulated output is $z=\begin{bmatrix} Cx \\ R^{1/2}u \end{bmatrix}$. The control input is the applied dc terminal voltage v_a. The control voltage limits are ± 24 volts.

In the mid equilibrium position, the state equation is (29). Using the algorithm in Section 2, the parameters of the algorithm are $C=I_4$, $R=I_4$, $\varepsilon=1$ and $\gamma=1500$. The MATLAB command of solving the algebraic Riccati equation is *care*. The solution is

$$X=\begin{bmatrix} 215.45 & 290.90 & 2.63 & 29.93 \\ 290.90 & 419.60 & 3.70 & 42.88 \\ 2.63 & 3.70 & 0.46 & 0.32 \\ 29.93 & 42.88 & 0.32 & 4.49 \end{bmatrix}$$

and the state feedback gain is

$$K=-\begin{bmatrix} 112.54 & 160.92 & 2.88 & 16.59 \end{bmatrix}.$$

The eigenvalues of the closed loop system are $\{-52.29,-9.37,-3.15\pm j4.43\}$. The experimental picture is shown in Fig.3. In the top equilibrium position, the state equation is (34). Using the algorithm in Section 2, the parameters of the algorithm are $C=I_4$, $R=I_4$, $\varepsilon=1$ and $\gamma=2500$. The solution is

$$X=\begin{bmatrix} 1340 & 1084 & 241.6 & 106.3 \\ 1084 & 898.8 & 195.3 & 87.5 \\ 241.6 & 195.3 & 43.7 & 19.2 \\ 106.3 & 87.5 & 19.2 & 8.6 \end{bmatrix}$$

and the state feedback gain is

$$K=\begin{bmatrix} 249.69 & 220.23 & 44.27 & 21.9 \end{bmatrix}.$$

The eigenvalues of the closed loop system are $\{-48.45,-15.49,-2.83,-6.33\}$. The experimental picture is shown in Fig. 4. The system responses are shown in Fig. 5 and Fig. 6. Figure 5 is the disturbance responses at mid position and figure 6 is the disturbance responses at top position. From initiation to 5 seconds, the pendulum is swung up by hands. When the pendulum is at the range of balance mode, the H_∞ state feedback control is initialized and the pendulum is kept upward. When the pendulum is swung upward, we tap on the pendulum. The responses of applied disturbances are shown in Fig. 5 and Fig. 6. The disturbances are attenuated quickly. The angle of the pendulum is maintained. From these experiments, the H_∞ state feedback control is guaranteed to reject the disturbance effectively.

6. Conclusions

This paper presents an application of H_∞ control. The H_∞ control is utilized to attenuate disturbances of pendubot. From the derived theoretic result, the derived feedback gain can minimize the infinite norm of system for some degree. The necessary work is to solve the algebraic Riccati equation in the proposed algorithm. For the experimental results, the H_∞ state feedback control is guaranteed to reject the disturbance effectively.

The pendubot is a benchmark system for under-actuated nonlinear systems. It is an impressive control system for the undergraduate control education. Hence, this paper is appropriate for the undergraduate curriculum. From these hands-on experiments, students can study the microcomputer and C-program courses. It also inspires students to study control theory curricula.

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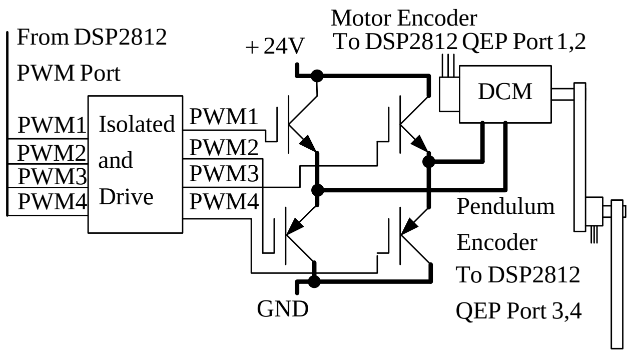

Fig. 1. The DSP-based pendubot schematic

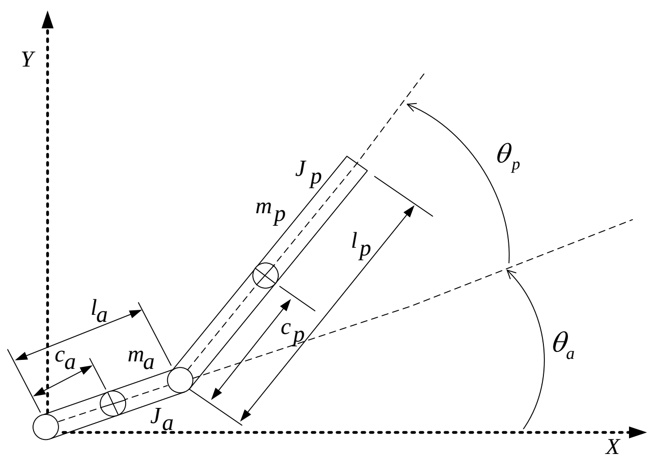

Fig. 2. The simplified pendubot schematic

Fig. 3.　Mid position

Fig. 4.　Top position

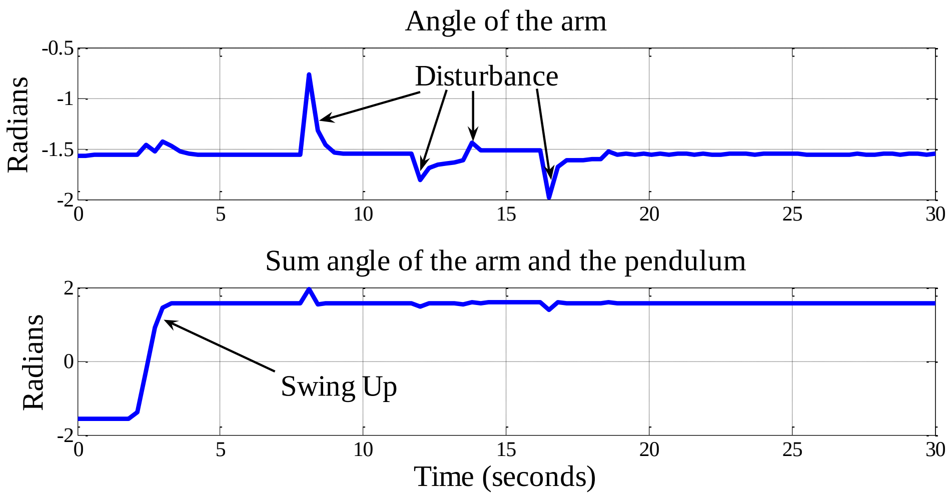

Fig. 5.　The disturbance responses at mid position

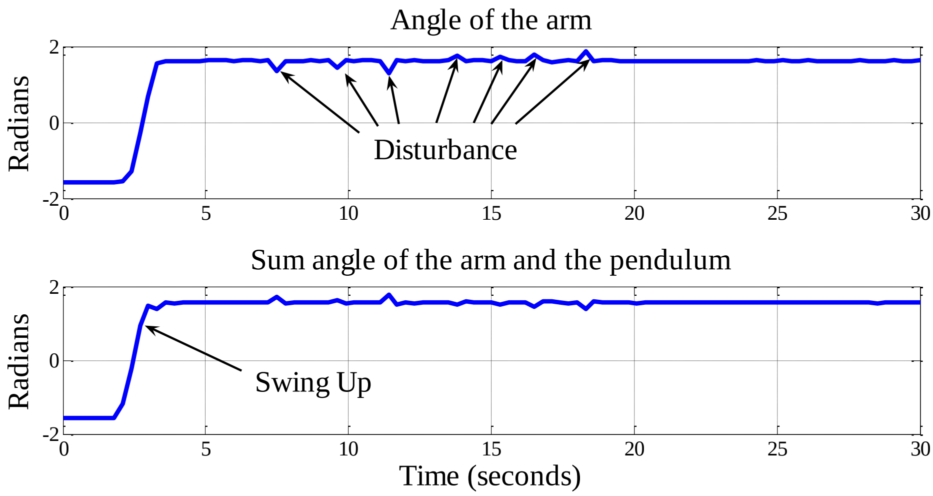

Fig. 6. The disturbance responses at top position

H_∞ State Feedback Disturbance Rejection of DSP-Based Pendubot

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Abstract

The disturbance rejection control of the DSP(Digital Signal Processor)-based pendubot is investigated in this paper. The H_∞ control method is utilized to reject disturbances and maintain the robust stability. There are two equilibrium positions of the pendubot, one is the middle position and the other is the top position. Both of these equilibrium positions are derived and designed. The pendubot is a kind of the inverted pendulums and an original nonlinear unstable system. The appropriate controls are necessary to maintain the robust stability. The pendubot is also an excellent educational tool and impresses students. The balancing control law used in this paper is the H_∞ state feedback controller. From the experimental results, the H_∞ controller is guaranteed to reject disturbances and maintain robust stability of the pendubot.

Keywords: H_∞ state feedback, Disturbance rejection, Pendubot, Digital signal processor.